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Section A : *Physique théorique.*

The theorem on Unitary Equivalence of Fock Representations

by

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ABSTRACT. — We prove that two Fock states ω_J and ω_K (not necessarily gauge invariant) on the CAR-algebra are unitarily equivalent if and only if $|J - K|$ is a Hilbert-Schmidt operator. We calculate explicitly the norm difference $\|\omega_J - \omega_K\|$.

Let (H, s) be a separable Euclidean space and J and K complex structures on (H, s), i. e.

$$J^+ = -J; \qquad J^2 = -1,$$
$$K^+ = -K; \qquad K^2 = -1.$$

Consider the operators

$$P = [J, K]_+; \qquad Q = [J, K]_-$$

and let $P = U|P|$, $Q = V|Q|$ be their polar decompositions, $|Q|$, $|P|$ and U commute with J and K; consequently the dimension of Ker P is even or infinite; Q is a normal operator, therefore V can be chosen such that $V^+ = -V$, $V^2 - = 1$. The same notations as in [1] are used : $\mathcal{A} = \overline{\mathcal{A}(H, s)}$ is the CAR-algebra and ω_J is any pure quasi-free state on $\mathcal{A}$; J satisfies : $J^+ = -J$, $J^2 = -1$.

THEOREM 1. — *Let the operator* P *be diagonalizable* [i. e. $\exists\, (\psi_i)_{i \in \mathbf{N}}$ orthonormal basis of H such that $P\psi_i = \mu_i \psi_i$, $\mu_i \in \mathbf{R}$ (reals)], *then there exists a family of subspaces* $(H_n)_{n \in \mathbf{N}}$ *of* H *invariant under* J *and* K *such that :*

(i) $H = \bigoplus_{n=0}^{\infty} H_n$;

(ii) dim H_0 *and* dim H_1 *is even or infinite,* dim $H_n = 4$ *for* $n \geq 2$;

(iii) $P = \sum_n \lambda_n p_n$, *where* $P_n H = H_n$; $\lambda_0 = -2$, $\lambda_1 = 2$ *and* $-2 < \lambda_n < 2$ *for* $n \geq 2$.

Proof. — Let $F = \text{Ker } Q$; F and $F^{\perp}$ (orthogonal complement of F for s) are invariant for J and K.

(*a*) Suppose $F^{\perp} = \{0\}$; then $JK = \frac{P}{2}$ is unitary and Hermitian, there exists a decomposition $F = H_0 + H_1$ such that $P = -P_0 + P_1$, where P_0 and P_1 are the orthogonal projection operators on H_0 respectively H_1, which are invariant under J and K and therefore dim H_0 and dim H_1 is even or infinite.

(*b*) Suppose $F = \{0\}$, let H_α be subspaces of H such that $PH_\alpha = \lambda_\alpha H_\alpha$. Because $[P, J]_- = [P, K]_- = 0$, the subspaces H_α are invariant for J and K. Remark that $P^2 + Q^+ Q = 4$, $Q^+ Q = |Q|^2$; therefore $|Q|$ has the same proper subspaces H_α as $|P|$. Let $|Q| H_\alpha = \mu_\alpha H_\alpha$, then $\lambda_\alpha^2 + \mu_\alpha^2 = 4$ for all α. Take any $\psi_\lambda \in H_\lambda$ and consider the subspaces H_{ψ_λ} generated by the real orthogonal set $\{\psi_\lambda, V\psi_\lambda, J\psi_\lambda, JV\psi_\lambda\}$. It is clear that H_{ψ_λ} is a real subspace invariant under J and K of dimension four.

In general $H = F + F^{\perp}$ the results of (*a*) and (*b*) prove the theorem.

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LEMMA. — *Let* π_J *and* π_K *be the Fock representations associated with* J *respectively* K. *If* π_J *and* π_K *are unitarily equivalent then* $[J, K]_+$ *has* -2 *as the only accumulation point of its spectrum.*

Proof. — Let $\{\psi_j\}_{j \in N}$ be any infinite orthonormal set of H and

$$L_n = \frac{-i}{n} \sum_{j=1}^{n} B(\psi_j) B(J\psi_j),$$

then

$$(\Omega_J, \pi_J(L_n)\Omega_J) = \omega_J(L_n) = 1.$$

Using Schwartz's inequality, we have

$$\|\pi_J(L_n)\Omega_J\| = 1 \qquad \text{furthermore} \qquad \left\| \left[\prod_{l=1}^{k} B(\psi_l), L_n \right]_- \right\| \leq \frac{k}{n}$$

proving

$$1 - \frac{k}{n} \leq \left\| \pi_J(L_n) \prod_{l=1}^{k} \pi_J(B(\psi_l))\Omega_J \right\| \leq 1 + \frac{k}{n},$$

i. e. $\pi_J(L_n)$ tends strongly to one for n tending to infinity. Because π_J and π_K are unitarily equivalent $\pi_K(L_n)$ tends strongly to one on $\mathcal{H}_K$ and therefore weakly.

Further the expression

$$\omega_K(L_n) = (\Omega_K, \pi_K(L_n)\Omega_K) = -\frac{1}{2n}\sum_{i=1}^{n} s(P\psi_i, \psi_i)$$

must tend to one for all orthonormal sets $(\psi_i)_{i\in N}$ which is possible if P has no accumulation points in its spectrum different from -2.

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THEOREM 2. — *If ω_J and ω_K are pure quasi-free states, then π_J and π_K are unitarily equivalent iff $|J - K|$ is a Hilbert-Schmidt operator.*

Proof. — By Theorem 1,

$$H = \bigoplus_{n=0}^{\infty} H_n; \qquad P = \sum_{n=0}^{\infty} \lambda_n P_n; \qquad P_n H = H_n,$$

where $\dim H_n = 4$ for $n \geq 2$. By the lemma, $\dim H_1 < \infty$. Let $\{\Phi_1, \ldots, \Phi_r; J\Phi_1, \ldots, J\Phi_r\}$ be an orthonormal basis of H_1 and

$$u_1 = \prod_{k=1}^{r} B(\Phi_k).$$

In each H_n $(n \geq 2)$ we choose the following orthonormal basis $(\psi_n, V\psi_n, J\psi_n, JV\psi_n)$, where ψ_n is any normalized vector of H_n and let

$$u_n = B(J\psi_n)\, B(\psi'_n),$$

where

$$\psi'_n = \frac{1}{(2-\lambda_n)^{\frac{1}{2}}}(J\psi_n + K\psi_n).$$

If u_0 is the unit of $\overline{\mathfrak{A}(H_0, s)}$, then for all $n \geq 0$ and all $x \in \overline{\mathfrak{A}(H_n, s)}$,

$$\omega_K(x) = \omega_J(u_n^* x u_n).$$

In order that $U = \bigotimes_{n=0}^{\infty} \pi_{J_n}(u_n)$ is an unitary operator on $\mathcal{H}_J = \bigotimes_{n=0}^{\infty} \mathcal{H}_{J_n}$ (J_n is the restriction of J to H_n) it is necessary and sufficient that

$$U\Omega_J \in \mathcal{H}_J \text{ i. e. } = \prod_{n=1}^{\infty} (\Omega_{J_n}, \pi_{J_n}(u_n)\Omega_{J_n}) = \prod_{n=1}^{\infty} \frac{1}{2}(2-\lambda_n)^{\frac{1}{2}}$$

does not vanish. But

$$\prod_{n=2}^{\infty} \frac{1}{2}(2-\lambda_n)^{\frac{1}{2}} \neq 0 \iff \prod_{n=2}^{\infty}\left(\frac{1}{2}-\frac{\lambda_n}{4}\right) \neq 0$$

$$\iff \frac{1}{4}\sum_{n=2}^{\infty}(2+\lambda_n) < \infty \quad \iff \mathrm{Tr}\,(2+\mathrm{P}) < \infty.$$

Otherwise $(\mathrm{J}-\mathrm{K})^{+}(\mathrm{J}-\mathrm{K}) = 2+\mathrm{P}$, therefore π_{J} and π_{K} are unitarily equivalent if $|\mathrm{J}-\mathrm{K}|$ is a Hilbert-Schmidt operator.

Conversely, suppose that $|\mathrm{J}-\mathrm{K}|$ is not a Hilbert-Schmidt operator, hence $\prod_{l=2}^{\infty}\left(\frac{1}{2}-\frac{\lambda_l}{4}\right)=0$. Let $\mathrm{E}_{n,m} = \bigoplus_{l=n}^{m} \mathrm{H}_l$; the restrictions of ω_{J} and ω_{K} to $\mathcal{A}(\mathrm{E}_{n,m}, s)$ remain pure states unitarily equivalent because if $\mathrm{U}_{n,m} = \prod_{l=n}^{m} u_l$, then

$$\forall\, x \in \mathcal{A}(\mathrm{E}_{n,m}, s), \quad \omega_{\mathrm{J}}(x) = \omega_{\mathrm{K}}(u_{n,m}\, x\, u^{*}_{n,m}) \quad [1].$$

Hence by Lemma 2.4 of [2]

$$\|(\omega_{\mathrm{J}}-\omega_{\mathrm{K}}) \,|\, \mathcal{A}(\mathrm{E}_{n,m}, s)\| = 2\left(1-|\omega_{\mathrm{J}}(u_{n,m})|^{2}\right)^{\frac{1}{2}}$$

$$= 2\left(1-\prod_{l=n}^{m}\left(\frac{1}{2}-\frac{\lambda_l}{4}\right)\right)^{\frac{1}{2}}.$$

Denote by $\mathcal{A}(\mathrm{E}_n, s)^{c}$ the commutant of $\mathcal{A}(\mathrm{E}_n, s)$ in $\mathcal{A}$. By lemma 2.3 of [2],

$$\|(\omega_{\mathrm{J}}-\omega_{\mathrm{K}}) \,|\, \mathcal{A}(\mathrm{E}_n, s)^{c}\| = \|(\omega_{\mathrm{J}}-\omega_{\mathrm{K}}) \,|\, \overline{\mathcal{A}(\mathrm{E}_n^{\perp}, s)}\|.$$

Since $\overline{\mathcal{A}(\mathrm{E}_n^{\perp}, s)}$ is the inductive limit of $\mathcal{A}(\mathrm{E}_{n,m}, s)$ when $m \to \infty$, we have

$$\|(\omega_{\mathrm{J}}-\omega_{\mathrm{K}}) \,|\, \mathcal{A}(\mathrm{E}_n, s)^{c}\| = \lim_{m \to \infty} \|(\omega_{\mathrm{J}}-\omega_{\mathrm{K}}) \,|\, \mathcal{A}(\mathrm{E}_{n,m}, s)\| = 2.$$

By lemma 2.1 of [2] π_{J} and π_{K} are not unitarily equivalent.

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COROLLARY. — *The representations π_{J} and π_{K} are unitarily equivalent if $\|\omega_{\mathrm{J}}-\omega_{\mathrm{K}}\| < 2$, and*

$$\|\omega_{\mathrm{J}}-\omega_{\mathrm{K}}\| = 2\left(1-\prod_{l=1}^{\infty}\left(\frac{1}{2}-\frac{\lambda_l}{4}\right)\right)^{\frac{1}{2}}.$$

Proof. — Lemma 2.1 of [2] proves that if π_J is not unitarily equivalent with π_K, then $\| \omega_J - \omega_K \| = 2$. Otherwise if π_J and π_K are equivalent, it follows from the calculations done in Theorem 2, that

$$\| \omega_J - \omega_K \| = 2\left(1 - \prod_{i=1}^{\infty} \left(\frac{1}{2} - \frac{\lambda_i}{4} \right) \right)^{\frac{1}{2}}.$$

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REFERENCES

[1] E. Balslev, J. Manuceau and A. Verbeure, *Commun. math. Phys.*, 8, 1968, p. 315.
[2] R. T. Powers and E. Størmer, *Commun. math. Phys.*, vol. 16, 1970, p. 1.

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