Translation Invariant States in Quantum Mechanics

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Abstract. We give a complete description of the states of the C.C.R. algebra for a finite number of degrees of freedom which are invariant with respect to subgroups of the translation group of phase space. We make precise some well-known results of quantum mechanics such as Bloch theorem.

§ I. Introduction

In this paper we shall deal with the algebra $\overline{\Delta(G, \xi)}$ of canonical commutation relations (C.C.R.) introduced in [2] (see also [3]) where G is an abelian group and ξ a bicharacter [Definition (2.1)]; for $G = \mathbb{R}^{2N}$ and $\xi((x, p), (x', p')) = \exp\left(-\frac{i}{2} \sum_{i=1}^{n} x_i p'_i - x'_i p_i\right)$, $(x, p) \in \mathbb{R}^{2N}$, $\overline{\Delta(\mathbb{R}^{2N}, \xi)}$ is the uniform closure of the $*$-algebra of the finite linear combinations of Weyl operators $\delta_{(x, p)}$. The interest of this algebra is that it has a large number of states; namely, any normalized linear positive function on the linear combinations of Weyl operators extends to a (unique) state of $\overline{\Delta(G, \xi)}$.

Translations by g in the phase space G are represented by a $*$-automorphism τ_g of $\overline{\Delta(G, \xi)}$ which is inner:

$$\tau_g \delta_{g'} = \delta_g \delta_{g'} \delta_g^{-1} = \xi(g, g')^2 \delta_{g'}, \quad g, g' \in G.$$

Let H be a subgroup of G and let H' be the set of elements g of G such that $\xi(g, h)^2 = 1$ for any $h \in H$. Then the invariant states ω of $\overline{\Delta(G, \xi)}$ with respect to the $\tau_h (h \in H)$ are those states for which $\omega(\delta_g) = 0$ if $g \notin H'$ [Proposition (2.13)]; conversely, let ω be a state of $\overline{\Delta(H', \xi')}$ where ξ' is the restriction of ξ to $H' \times H'$; then there exists an invariant extension $\bar{\omega}$ of ω to $\overline{\Delta(G, \xi)}$ which is given by $\bar{\omega}(\delta_g) = 0$ if $g \notin H'$ [Proposition (2.14)]. Moreover if $\overline{\Delta(H', \xi')}$ is abelian and if ω is pure, $\bar{\omega}$ is the unique (hence pure) extension of ω to $\overline{\Delta(G, \xi)}$ if and only if $\overline{\Delta(H', \xi')}$ is maximal abelian [Proposition (2.18)].

In Section 3, we specialize $\overline{\varDelta(\mathbb{R}^2, \xi)}$, and show that we only need consider closed subgroups of phase space as invariance subgroups [Lemma (2.11)]. Hence except for the trivial cases which are easy to handle, and up to isomorphisms there are three possible cases for ξ abelian subgroups (i.e. for which $\boldsymbol{H} \subset \boldsymbol{H'}$);

i) The subgroup of $\mathbb{R}^2$ is $\mathbb{R} \times \{0\}$; invariant pure states ω of $\overline{\varDelta(\mathbb{R}^2, \xi)}$ for which $x \to \omega(\delta_{x,p})$ is continuous are of the form

$$\omega_{p_0}(\delta_{x,p}) = 0 \quad \text{if} \quad p \neq 0$$
$$\omega_{p_0}(\delta_{x,p}) = \exp(i p_0 x) \quad p_0 \in \mathbb{R}.$$

These states are vector states of a representation of $\overline{\varDelta(\mathbb{R}^2, \xi)}$ in a non separable Hilbert space [Theorem (3.6)]. These are the analogues of plane waves of the usual formalism.

ii) The subgroup of $\mathbb{R}^2$ is $\mathbb{Z} a \times \{0\}$ where $\mathbb{Z}$ is the group of integers, $a \in \mathbb{R}$. Invariant pure states of $\overline{\varDelta(\mathbb{R}^2, \xi)}$ for which $x \to \omega(\delta_{x,p})$ is continuous are of the form

$$\omega(\delta_{x,p}) = 0 \quad \text{if} \quad p \neq \frac{2\pi m}{a}$$

$$\omega\left(\delta_{x, \frac{2\pi m}{a}}\right) = e^{iqx} e^{\frac{i\pi m x}{a}} \sum_{n \in \mathbb{Z}} \overline{f(n)} \, e^{\frac{2i\pi n x}{a}} \, f(n+m)$$

$$q \in \left[0, \frac{2\pi}{a}\right[\qquad f \in l_2(\mathbb{Z})$$

[Bloch theorem (3.13)]. These states are vector states of the previous representation.

iii) The subgroup of $\mathbb{R}^2$ is $\mathbb{Z} a \times \mathbb{Z} \frac{2\pi}{a}$, $a \in \mathbb{R}$. Invariant pure states ω of $\overline{\varDelta(\mathbb{R}^2, \xi)}$ are of the form

$$\omega_{\beta\gamma}(\delta_{x,p}) = 0 \quad \text{if} \quad (x, p) \neq \left(n\lambda, \frac{2\pi m}{\lambda}\right)$$

$$\omega_{\beta\gamma}\left(\delta_{n\lambda, \frac{2\pi m}{\lambda}}\right) = \exp\left(i\pi mn + i\beta n\lambda + i\gamma \frac{2\pi m}{\lambda}\right)$$

$$n, m \in \mathbb{Z}, \qquad \beta \in [0, 2\pi[, \qquad \gamma \in [0, \lambda[\, .$$

They are vector states of a unique representation within a non-separable Hilbert space; this representation is disjoint from the previous one [Proposition (3.23)].

Finally we study the action of the group of linear non singular transformations of phase space on the above states and give the conditions of implementation of these transformations [Proposition (3.27)].

§ II. Definitions and Elementary Results

Most of the results of this section are slight generalizations of the corresponding results of Ref. [3]; hence they are in general given without proofs (see also [4]).

The bicharacters we shall define now are of central importance in what follows:

Definition (2.1). Let G be an abelian group: a bicharacter ξ is a function $\xi : G \times G \to T$ (the group of complex numbers of modulus one) such that:

$$\xi(g, g') = \overline{\xi(g', g)} \qquad \forall\, g, g' \in G \tag{2.2}$$

$$\xi(g, g_1 + g_2) = \xi(g, g_1)\,\xi(g, g_2) \qquad \forall\, g, g_1, g_2 \in G. \tag{2.3}$$

A similar procedure to that in [3] can be used to construct a $*$-algebra $\Delta(G, \xi)$ for ξ a bicharacter.

Definition (2.4). $\Delta(G, \xi)$ where G is an abelian group and ξ is a bicharacter is the set of finite linear combinations[1]:

$$a = \sum_{i=1}^{N} \lambda_i \delta_{g_i} \qquad \lambda_i \in \mathbb{C},\, g_i \in G$$

endowed with the following product and $*$-operation:

$$\delta_g \delta_h = \xi(g, h)\, \delta_{g+h} \tag{2.5}$$

$$\delta_g^* = \xi(g, g)\, \delta_{-g}. \tag{2.6}$$

As shown in [3] there exists on $\Delta(G, \xi)$ a completely regular norm (i.e. one satisfying $\|a^* a\| = \|a\|^2$)

$$\|a\| = \sup_{\omega \in \xi_G} \sqrt{\omega(a^* a)} \qquad a \in \Delta(G, \xi) \tag{2.7}$$

where ξ_G is the set of positive normalized forms on $\Delta(G, \xi)$.

Moreover it is the unique completely regular norm on $\Delta(G, \xi)$ when ξ is non-degenerate (see below for definition).

We denote by $\overline{\Delta(G, \xi)}$ the completion of $\Delta(G, \xi)$ with respect to this norm.

Any positive linear form (resp. $*$-representation) of $\Delta(G, \xi)$ extends to a positive linear form (resp. $*$-representation) of $\overline{\Delta(G, \xi)}$ ([3], 3.4).

Let us mention the following lemma:

Lemma (2.8). *Let G be an abelian group, H a subgroup, ξ a bicharacter on G; denote also by ξ the restriction of ξ to $H \times H$; then*

$$\overline{\Delta(H, \xi)} \subseteq \overline{\Delta(G, \xi)}.$$

Moreover if $H \subsetneq G$, then $\overline{\Delta(H, \xi)} \subsetneq \overline{\Delta(G, \xi)}$.

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[1] Notice that the unitary elements δ_g are linearly independent (see [2], 2.2.2).

We shall need in the sequel of the notion of commutant of a subgroup of G. So we define

Definition (2.9). Let H be a subgroup of G. We define H', the commutant of H, as

$$H' = \{g \in G; \; \xi(g, h)^2 = 1 \;\; \forall\, h \in H\}.$$

H will be ξ-abelian if and only if $H \subseteq H'$ and maximal ξ-abelian if and only if $H = H'$.

We shall say that ξ is non-degenerate if and only if $G' = \{0\}$.

One can easily verify the following properties of the commutant operation:

Lemma (2.10). *Let G be an abelian group; ξ a bicharacter, H, H_1 and H_2 subgroups of G. Then:*
   i. H' *is a subgroup.*
   ii. $H \subseteq H''$.
   iii. $H' = H'''$.
   iv. $H_1 \subseteq H_2 \Rightarrow H_2' \subseteq H_1'$.

The next lemma will be of some importance in the last section.

Lemma (2.11). *Let G be an abelian topological group; let ξ be a bicharacter such that the function $g \to \xi(g, g')$ is continuous $\forall\, g' \in G$.*

Then if H is a subgroup of G, $H' = (\bar{H})'$ where $\bar{H}$ denotes the closure of H.

Finally we denoted by τ the canonical injection of the group G into the inner $*$-automorphisms of $\overline{\Delta(G, \xi)}$

$$\tau_g(\delta_h) = \xi(g, h)^2\, \delta_h. \tag{2.12}$$

An easy generalization of the previous results can be made for a general group (non-abelian), but in view of the applications of the last section and of the simplification of the notations we just mention it.

Now we shall give some easy results about extensions of states on subalgebras. Proofs are standard or even simplified by the topology (viz. the discrete topology chosen on G) so in general they are not given.

Lemma (2.13). *Let G be an abelian group, ξ a bicharacter, H a subgroup of G, τ the canonical injection of H into the $*$-automorphisms group of $\overline{\Delta(G, \xi)}$; then if ω is a state of $\overline{\Delta(G, \xi)}$,*

$$\omega \circ \tau_h = \omega \quad \forall\, h \in H \Rightarrow \omega(\delta_g) = 0 \quad \forall\, g \notin H'.$$

The next result ensures the existence of an invariant state.

Theorem (2.14). *Let G be an abelian group, ξ a bicharacter, H a subgroup of G; let ω be a positive linear form on $\Delta(H, \xi)$; then $\bar{\omega}$ defined as*

$$\bar{\omega}(\delta_h) = \omega(\delta_h) \quad \forall \, h \in H$$

$$\bar{\omega}(\delta_g) = 0 \qquad \forall \, g \notin H$$

gives a positive linear form on $\Delta(G, \xi)$.

Using the remark before Lemma (2.8) one has the following corollary.

Corollary (2.15). *Any state ω of the subalgebra $\overline{\Delta(H, \xi)}$ of $\overline{\Delta(G, \xi)}$ can be extended to a state $\bar{\omega}$ of $\overline{\Delta(G, \xi)}$ in the following way*

$$\bar{\omega}(a) = \omega(a) \quad \text{if} \quad a \in \overline{\Delta(H, \xi)}$$

$$\bar{\omega}(\delta_g) = 0 \qquad \text{if} \quad g \notin H .$$

The next theorem gives an explicit realization of the representation associated to the extension $\bar{\omega}$.

Theorem (2.16). *Let G be an abelian group, ξ a bicharacter and H a subgroup of G. Let ω be a state of $\overline{\Delta(H, \xi)}$, $(\mathcal{H}_\omega, \Omega_\omega, \pi_\omega)$ the corresponding Gelfand-Naimark-Segal (G.N.S.) triplet. Let $\bar{\omega}$ be the extension of ω to $\overline{\Delta(G, \xi)}$ previously described. Let S be an application of G into G such that*

$$S(0) = 0$$

$$S(g + h) = S(g) \quad \forall \, g \in G, \quad \forall \, h \in H$$

$$g - S(g) = T(g) \in H \quad \forall \, g \in G .$$

Let $\mathcal{H}_{\bar{\omega}}$ be the Hilbert space $l_2(G/H) \otimes \mathcal{H}_\omega$, $[g]$ the function of $l_2(G/H)$ which is one on the class of g and zero otherwise.

Then $\bar{\pi}$ defined as the linear transformation such that

$$\bar{\pi}(\delta_k) [S(g) \otimes \pi_\omega(\delta_{T(g)}) \Omega_\omega = \Xi(k, g) [S(g + k)] \otimes \pi_\omega(\delta_{T(g+k)}) \Omega_\omega$$

where

$$\Xi(k, g) = \xi(k, g) \, \xi(S(g), T(g)) \, \overline{\xi(S(g + k), T(g + k))}$$

extends to a $*$-representation of $\overline{\Delta(G, \xi)}$ in $l^2(G/H) \otimes \mathcal{H}_\omega$ with cyclic vector $[0] \otimes \Omega_\omega$; and it is unitarily equivalent to the representation constructed from $\bar{\omega}$ by the usual G.N.S. procedure.

First notice that $[S(g)] \otimes \pi_\omega(\delta_h) \Omega_\omega = [S(g')] \otimes \pi_\omega(\delta_{T(g')}) \Omega_\omega$ where $g' = S(g) + h, \; g \in G, \; h \in H$.

Hence $\bar{\pi}(\delta_k)$ is defined over a dense set of vectors; moreover $\bar{\pi}$ is well-defined since if

$$\Phi = [S(g)] \otimes \left(\sum_{i=1}^{n} \lambda_i \pi_\omega(\delta_{h_i}) \right) \Omega_\omega = 0$$

then

$$\bar{\pi}(\delta_k) \, \Phi = \Xi(k, S(g)) \, [S(g+k)] \otimes \pi_\omega(\delta_{T(S(g)+k)}) \left(\sum_{i=1}^n \lambda_i \pi_\omega(\delta_{h_i}) \right) \Omega_\omega$$
$$= 0 \, ;$$

the representation character of $\bar{\pi}$ follows from an easy but tedious calculation.

Corollary (2.17). $\bar{\pi}$ *is reducible if* π_ω *is reducible.*

The restriction of $\bar{\pi}$ to $\overline{\Delta(H, \xi)}$ is given as one can easily compute by

$$\bar{\pi}(\delta_h) \, [S(g)] \otimes \pi_\omega(\delta_{T(g)}) \, \Omega_\omega = \xi(h, S(g))^2 \, [S(g)] \otimes \pi_\omega(\delta_h) \, \pi_\omega(\delta_{T(g)}) \, \Omega_\omega$$

for $h \in H$.

The next results are connected with the uniqueness and irreducibility of $\bar{\omega}$.

Theorem (2.18). *Let* G *be an abelian group and* ξ *a bicharacter. Let* H *be a subgroup of* G *such that* $H \subset H'$; *then if* ω *is a pure state of* $\overline{\Delta(H, \xi)}$, $\bar{\omega}$ *is the unique extension of* ω *to* $\overline{\Delta(G, \xi)}$ *if and only if* $H = H'$, *and consequently is pure.*

Proof. Since $\overline{\Delta(H, \xi)}$ is abelian, ω is a character, hence

$$|\omega(\delta_h)| = 1 \, .$$

Moreover let $\bar{\omega}$ be an extension of ω to $\overline{\Delta(G, \xi)}$ [see Corollary (2.15)] then:

$$\bar{\omega}(\delta_g \delta_h \delta_g^*) = \xi(g, h)^2 \, \omega(\delta_h) \, .$$

Let $(\bar{\Omega}, \bar{\pi}, \mathcal{H})$ be the G.N.S. triplet associated to $\bar{\omega}$; then from the previous relation

$$\bar{\pi}(\delta_h) \, \bar{\pi}(\delta_g^*) \, \bar{\Omega} = \xi(g, h)^2 \, \omega(h) \, \bar{\pi}(\delta_g^*) \, \bar{\Omega} \, ,$$

hence

$$\bar{\omega}(\delta_g) = (\bar{\Omega} \,|\, \bar{\pi}(\delta_g) \, \bar{\Omega})$$

$$= \overline{\omega(h)} \, (\bar{\Omega} \,|\, \bar{\pi}(\delta_g) \, \bar{\pi}(\delta_h) \, \bar{\Omega})$$

$$= \xi(g, h)^2 \, (\bar{\Omega} \,|\, \bar{\pi}(\delta_g) \, \bar{\Omega}) \, ;$$

hence the result. Notice that from ([6], 2.10.2) there exists at least one pure extension of a pure state.

From the previous construction we have the following result:

Corollary (2.19). *The representation space corresponding to* $\bar{\omega}$ *is separable if and only if* $\mathcal{H}_\omega$ *is separable and* G/H *is at most countable.*

§ III. Applications: Invariant States of the C.C.R. of a Finite Number of Degrees of Freedom with Respect to a ξ-Abelian Subgroup of the Phase Space

In the following we shall be interested in $G = \mathbb{R}^{2N}$ considered as $\mathbb{R}^N \times \mathbb{R}^N$; a typical element of $\mathbb{R}^{2N}$ will be written as

$$X = (x_1, \ldots, x_N, p_1, \ldots, p_N) \quad x_i, p_i \in \mathbb{R}.$$

Moreover we shall deal with the following bicharacter on $\mathbb{R}^{2N}$

$$
\begin{aligned}
\xi((x_i, p_i), (x_i', p_i')) &= \exp\left(-\frac{i}{2}\sigma(X, X')\right) \\
&= \exp\left(-\frac{i}{2}(X \mid JX')\right) = \exp\left(-\frac{i}{2}\sum_{i=1}^{N} x_i p_i' - x_i' p_i\right)
\end{aligned}
\tag{3.1}
$$

where $(X \mid X')$ denotes the euclidean scalar product on $\mathbb{R}^{2N}$ and σ and J have obvious definitions:

$$J(x_i, p_i) = (p_i, -x_i). \tag{3.2}$$

This bicharacter is continuous with respect to the natural topology of $\mathbb{R}^{2N}$ hence by (2.11) we only shall consider closed subgroups of $\mathbb{R}^{2N}$.

For the sake of physical applications we have in mind, we shall consider only ξ-abelian subgroups of $\mathbb{R}^{2N}$; moreover we shall treat only the case $N = 1$; the general case can be dealt with by the following lemma which uncouples the different degrees of freedom.

Lemma (3.3). *Let Γ be a lattice in $\mathbb{R}^{2N}$ and Γ' its reciprocal lattice, viz. Γ is of the form*

$$a \in \Gamma \Leftrightarrow a = \sum_{i=1}^{N} m_i a_i \quad m_i \in \mathbb{Z}$$

$$a' \in \Gamma' \Leftrightarrow a' = \sum_{i=1}^{N} n_i \alpha_i \quad n_i \in \mathbb{Z}$$

with

$$\sigma(a_i, a_j) = \sigma(\alpha_i, \alpha_j) = 0 \quad \forall i, j = 1 \ldots N$$

$$\sigma(a_i, \alpha_j) = 2\pi \delta_{ij} \quad \forall i, j = 1 \ldots N$$

($\Gamma \cup \Gamma'$ defines the most general ξ-abelian discrete subgroup of $\mathbb{R}^{2N}$). Then there exists a linear non-singular transformation M such that

$$\sigma(MX, MY) = \sigma(X, Y) \quad X, Y \in \mathbb{R}^{2N}$$

$$(Ma_i) = \Big(0, \ldots 1, \ldots 0, \underbrace{0, \ldots 0}_{N \text{ times}}\Big)$$
$$_{i\text{-th place}}$$

$$(M\alpha_i) = \Big(\underbrace{0, \ldots 0}_{N \text{ times}}, 0, \ldots 2\pi, \ldots 0\Big).$$
$$_{i\text{-th place}}$$

The existence of M is ensured by the isomorphism of symplectic spaces of the same finite dimension. For the sake of practical applications it is worth while to give an explicit form of M

$$M_{ij} = -(J a_i)_j \quad \text{if} \quad i > N$$

$$M_{ij} = \frac{1}{2\pi} (J \alpha_i)_j \quad \text{if} \quad i \leq N. \tag{3.4}$$

Classification of closed subgroups of $\mathbb{R}^2$ is well-known; the next table give the ones for which the restriction of ξ is ± 1, except for the trivial subgroup.

Subgroup	ξ-commutant	Interpretation of H	Property of H'	Corresponding invariant state
$\mathbb{R} \times \{0\}$	$H' = H'' = H$	Translations	Max-abelian	Plane waves
$\mathbb{Z} \times \{0\}$	$H' = \mathbb{R} \times \mathbb{Z}$ $H'' = H$	Discrete translations	Contains $\mathbb{R} \times \{0\}$ which is max-abelian	Bloch waves
$\mathbb{Z} \times \mathbb{Z}$	$H' = H'' = H$		Max-abelian	Zak waves

We shall study these three cases separately.

1st Case: Plane Waves

Proposition (3.5). *Let $\{\tau_a ; a \in \mathbb{R}\}$ be the group of $*$-automorphisms of $\overline{\Delta(\mathbb{R}^2, \xi)}$ such that*

$$\tau_a \delta_{x,p} = e^{-iap} \delta_{x,p} \quad \forall a \in \mathbb{R}.$$

Every pure state ω of $\overline{\Delta(\mathbb{R}^2, \xi)}$ such that

$$x \to \omega(\delta_{x,p}) \quad \text{is continuous}$$

and invariant with respect to $\tau_a (\forall a \in \mathbb{R})$ is a state ω_{p_0} defined by

$$\omega_{p_0}(\delta_{x,p}) = 0 \quad \text{if} \quad p \neq 0$$

$$= e^{ip_0 x} \quad \text{if} \quad p = 0.$$

The proof is obvious and we shall give in the next proposition the structure of the representation.

Proposition (3.6). *Let ω_{p_0} be a state defined as*

$$\omega_{p_0}(\delta_{x,p}) = 0 \quad \text{if} \quad p \neq 0$$

$$= e^{ip_0 x} \quad \text{if} \quad p = 0.$$

Let $\mathcal{H}_{p_0}$, π_{p_0}, and Ω_{p_0} be the canonical Hilbert space, representation and cyclic vector constructed via the G.N.S. theorem. Then

   i. $\mathcal{H}_{p_0} = l^2(\mathbb{R})$ is not separable.

   ii. π_{p_0} is irreducible.

   iii. $\forall \, p' \in \mathbb{R}$ $\omega_{p'}$ is a vector state of π_{p_0}.

   iv. π_{p_0} is disjoint from any Weyl state.

Proof. i. Is obvious from Theorem (2.16); ii. is clear from (2.18); to prove iii., let us consider $|q\rangle \in l_2(\mathbb{R})$ such that $|q\rangle(x) = 0$ if $x \neq q$ and $|q\rangle(x) = 1$ if $x = q$. Then let S and T be:

$$S(x, p) = (0, p)$$

$$T(x, p) = (x, 0).$$

Consequently according to Theorem (2.16)

$$\pi_{p_0}(\delta_{x,p}) |q\rangle \otimes \Omega_{p_0} = e^{-\frac{1}{2}(p + 2q)x} e^{i p_0 x} |q + p\rangle \otimes \Omega_{p_0}. \tag{3.7}$$

Hence

$$\omega_{p'}(\delta_{x,p}) = (|p_0 - p'\rangle \otimes \Omega_{p_0} | \pi_{p_0}(\delta_{x,p}) |p_0 - p'\rangle \otimes \Omega_{p_0}). \tag{3.8}$$

Moreover ω_{p_0} is disjoint from any Weyl state; indeed if it is not disjoint it is unitarily equivalent to any Weyl pure state, since it is pure, but the usual Schrödinger representation acts within a separable Hilbert space.

Proposition (3.9). *Let ω_{p_0} be as previously.*

   I. $\tau_a(\forall \, a \in \mathbb{R})$ *is unitarily implemented within π_{p_0} by $\pi_{p_0}(\delta_{a,0})$ which is strongly continuous in a; its spectrum is purely discrete and is the set* $\{\exp(i p' a); p' \in \mathbb{R}\}$.

   II. *Let $\{\alpha_t\}_{t \in \mathbb{R}}$ be the group of $*$-automorphisms of $\overline{\Delta(\mathbb{R}^2, \xi)}$ such that*

$$\alpha_t \delta_{x,p} = \delta_{x + \frac{t}{m} p, p} \quad m > 0, \quad t \in \mathbb{R};$$

ω_{p_0} *is α_t invariant; α_t is unitarily implemented within π_{p_0} by a strongly continuous group of unitaries whose infinitesimal generator has a discrete spectrum from* $-\dfrac{p_0^2}{2m}$ *to ∞.*

Proof. I. Is obvious. In order to prove II we have that

$$(\Omega_{p_0} | \pi_{p_0}(\delta_{x_1, p_1}) \, \pi_{p_0}(\alpha_t \delta_{x_2, p_2}) \, \Omega_{p_0})$$

$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad p_1 \neq -p_2 \tag{3.10}$$

$$= \exp\left(i(x_1 + x_2)\left(p_0 + \frac{p}{2}\right) - \frac{it}{m}\left(p_0 p + \frac{p^2}{2}\right)\right) \quad \text{if} \quad p_1 = -p_2 = p$$

which shows that $\{U_t, t \in \mathbb{R}\}$ the group of unitary operators which implements α_t is continuous; moreover that its spectrum extends from $-\dfrac{p_0^2}{2m}$ to ∞.

2nd Case: Bloch Waves

In this case we shall deal with discrete translations; the invariance group Γ is generated by a $*$-automorphism τ_a such that

$$\tau_a \delta_{x,p} = e^{-iap} \delta_{x,p}.$$

Hence

$$\Gamma = \{(na, 0); n \in \mathbb{Z}\}. \tag{3.11}$$

A classical theorem [7], due to Bloch, states that if the hamiltonian of one particle is invariant by discrete translations of the type na, $n \in \mathbb{Z}$, one can find eigen-solutions of the Schrödinger equation as

$$\psi(x) = e^{iqx} \mathcal{U}_q(x) \tag{3.12}$$

where $q \in \left[0, \dfrac{2\pi}{a}\right[$ and $\mathcal{U}_q(x + a) = \mathcal{U}_q(x)$. Let us state the same result in our formalism.

Theorem (Bloch) (3.13). *Any pure state ω of $\overline{\Delta(\mathbb{R}^2, \xi)}$, invariant with respect to Γ and such that $x \to \omega(\delta_{x,p})$ is continuous at $x = 0$ is of the form*

$$\omega(\delta_{x,p}) = 0 \quad \text{if} \quad p \neq \frac{2\pi m}{a}, \quad m \in \mathbb{Z}$$

$$\omega\left(\delta_{x, \frac{2\pi m}{a}}\right) = e^{iqx} e^{\frac{i\pi mx}{a}} \sum_{n \in \mathbb{Z}} \overline{f(n)} \, e^{\frac{2i\pi nx}{a}} f(n+m)$$

where $q \in \left[0, \dfrac{2\pi}{a}\right[$ and f is in $l_2(\mathbb{Z})$ with $\|f\|_2 = 1$.

We shall prove the necessity of the previous formula through a number of lemmas. The sufficiency will be much easier to prove.

Lemma (3.14). *Let ω be a pure state of $\overline{\Delta(\mathbb{R}^2, \xi)}$ invariant with respect to Γ; $\omega\!\restriction$, its restriction to $\overline{\Delta(\Gamma', \xi)}$, is still a pure state.*

$$\Gamma' = \left\{\left(x, \frac{2\pi m}{a}\right); \; m \in \mathbb{Z}\right\}.$$

According to Lemma (2.13)

$$\omega(\delta_{x,p}) = 0 \quad \text{if} \quad (x, p) \notin \Gamma'.$$

Moreover if $\omega\!\restriction$ is not pure it has a non-trivial decomposition

$$\omega\!\restriction = \alpha \varrho_1 + (1 - \alpha) \varrho_2$$

and $\omega = \overline{\omega}\!\restriction$ has a similar decomposition

$$\omega = \alpha \overline{\varrho}_1 + (1 - \alpha) \overline{\varrho}_2$$

since the invariant extension of $\omega\!\restriction$ is given by Lemma (2.15).

Lemma (3.15). *Let ω be a pure state of $\overline{\Delta(\Gamma', \xi)}$. Then there exists*

$$q \in \left[0, \frac{2\pi}{a}\right[\quad \text{such that} \quad \varrho\left(\tilde{\delta}_{y, \frac{2\pi m}{a}}\right) = e^{-2iqy}\, \omega\left(\delta_{iy, \frac{2\pi m}{a}}\right) \quad \text{for any} \quad y \in [0, a[$$

is a pure state of the algebra $\overline{\Delta(T \times \mathbb{Z}, \zeta)}$ with $\varrho(\tilde{\delta}_{\frac{a}{2}, 0}) = 1$. Here

$T \times \mathbb{Z} = \left\{\left(y, \frac{2\pi m}{a}\right); \ y \in [0, a[, m \in \mathbb{Z}\right\}$ *is the abelian group of the torus*

multiplied by integers (with its natural topology) and

$$\zeta : (T \times \mathbb{Z}) \times (T \times \mathbb{Z}) \to T$$

defined by

$$\zeta\left(\left(y, \frac{2\pi m}{a}\right), \left(y', \frac{2\pi m'}{a}\right)\right) = \exp\left(-\frac{2i\pi}{a}(m'y - my')\right)$$

is continuous bicharacter of $T \times \mathbb{Z}$.

Conversely, if ϱ is a (pure) state of the algebra $\overline{\Delta(T \times \mathbb{Z}, \zeta)}$ with $\varrho\left(\tilde{\delta}_{\frac{a}{2}, 0}\right) = 1$, then $\omega\left(\delta_{y, \frac{2\pi m}{a}}\right) = e^{iqy} \varrho\left(\tilde{\delta}_{\frac{y}{2}, \frac{2\pi m}{a}}\right)$ is a (pure) state of $\overline{\Delta(\Gamma', \xi)}$.

This lemma is of central importance in the proof of Theorem (3.13). Let us define q. Since ω is pure, it is a character when restricted to the center of $\overline{\Delta(\Gamma', \xi)}$ which is generated by the $\delta_{na, 0}, n \in \mathbb{Z}$.

Hence there exists $q \in \left[0, \frac{2\pi}{a}\right[$ such that

$$\omega(\delta_{na, 0}) = \exp(-iqna) \quad n \in \mathbb{Z}. \tag{3.16}$$

According to [3], Proposition (3.4) we have only to prove the ζ-positivity of ϱ; given $\lambda_i \in \mathbb{C}$, $y_i \in [0, a[, m_i \in \mathbb{Z} \ i = 1 \ldots N$

$$\begin{aligned} & u_i \lambda_j \sum_{i,j=1}^{N} \bar{\lambda}_i \lambda_j \varrho\left(\tilde{\delta}_{-y_i + a\delta_i, \frac{-2\pi m_i}{a}} \tilde{\delta}_{y_j, \frac{2\pi m_j}{a}}\right) && \begin{aligned} \delta_i &= 0 \quad \text{if} \quad y_i = 0 \\ &= 1 \quad \text{if} \quad y_i > 0 \end{aligned} \\ &= \sum_{i,j=1}^{N} \bar{\lambda}_i \lambda_j \exp\left(\frac{2i\pi}{a}(y_i m_j - y_j m_i)\right) && \begin{aligned} \delta_{ij} &= 0 \quad \text{if} \quad y_j - y_i + a\delta_i < a \\ &= 1 \quad \text{if} \quad y_j - y_i + a\delta_i \geqq a \end{aligned} \\ & \quad \cdot \varrho\left(\tilde{\delta}_{-y_i + y_j + a\delta_i - a\delta_{ij}, \frac{2\pi}{a}(m_j - m_i)}\right) \\ &= \sum_{i,j=1}^{N} \bar{\lambda}_i \lambda_j \exp\left(\frac{2i\pi}{a}(y_i m_j - y_j m_i) - 2iq(-y_i + y_j + a\delta_i - a\delta_{ij})\right) \\ & \quad \cdot \omega\left(\delta_{-2(y_i - y_j - a\delta_i + a\delta_{ij}), \frac{2\pi}{a}(m_j - m_i)}\right); \end{aligned}$$

using (3.16) the previous expression is reduced to

$$\sum_{i,j=1}^{N} \bar{\lambda}_i \lambda_j e^{-2iq(y_j - y_i)} \exp\left(\frac{2i\pi}{a}(m_j y_i - m_i y_j)\right)$$

$$\cdot \omega\left(\delta_{-2y_i + 2y_j, \frac{2\pi}{a}(m_j - m_i)}\right) \geqq 0$$

by positivity of ω.

By a similar computation, it is easy to prove that if ϱ is a state of $\overline{\varDelta(T \times \mathbb{Z}, \zeta)}$ with $\varrho(\tilde{\delta}_{\frac{a}{2}, 0}) = 1$, then ω is a state of $\overline{\varDelta(\varGamma', \xi)}$. Then it follows that ϱ is pure if and only if ω is pure.

Lemma (3.17). *Let ω be a pure state of $\overline{\varDelta(\varGamma', \xi)}$; then ϱ, defined in Lemma (3.15), is continuous with respect to the natural topology of $T \times \mathbb{Z}$ if and only if $x \rightarrow \omega\left(\delta_{x, \frac{2\pi m}{a}}\right)$ is continuous at $x = 0$.*

The proof is obvious.

Lemma (3.18). *Let ϱ be a pure state of $\overline{\varDelta(T \times \mathbb{R}, \zeta)}$ continuous with respect to the natural topology of $T \times \mathbb{Z}$ and such that $\varrho(\tilde{\delta}_{\frac{a}{2}, 0}) = 1$; then there exists $f \in l_2(\mathbb{Z})$, $\|f\|_2 = 1$ such that*

$$\varrho\left(\tilde{\delta}_{y, \frac{2\pi m}{a}}\right) = e^{\frac{2i\pi m y}{a}} \sum_{n \in \mathbb{Z}} \overline{f(n)} \, e^{\frac{4i\pi n y}{a}} \, f(n+m).$$

This lemma follows from Theorem 1 of [9] (also see [8, 10]). Collecting these results one can see that the formula in Theorem (3.13) is a necessary condition; the sufficiency is almost obvious.

The correspondence of our result to the usual one is obvious; the $f(n)$ in Theorem (3.13) are nothing but the Fourier coefficients of $\mathcal{U}_q(x)$ in (3.12).

The states we have defined in Proposition (3.5) are invariant with respect to any translation, hence by $\varGamma$; actually we have the following result:

Proposition (3.19). *Let ω be a pure state of $\overline{\varDelta(\mathbb{R}^2, \xi)}$ invariant with respect to $\varGamma$ and such that $x \rightarrow \omega(\delta_{x,p})$ is continuous at $x = 0$; it is a vector state of π_{p_0} defined in Proposition (3.6). More precisely*

$$\omega(\delta_{x,p}) = (\Phi \,|\, \pi_{p_0}(\delta_{x,p}) \, \Phi)$$

where

$$\Phi = \sum_n f(n) \, \pi_0\left(\delta_{0, -\left(q + \frac{2\pi n}{a}\right)}\right) \Omega_0.$$

The proof requires an obvious calculation.

3rd Case: Zak Waves

As we saw previously there is a third type of closed ξ-abelian subgroups of $(\mathbb{R}^2, \xi)$; namely those isomorphic to $\mathbb{Z}^2$; the corresponding lattice is generated by two vectors $a = (a_1, a_2)$ and $\alpha = (\alpha_1, \alpha_2)$, and the abelianness of the generated group implies that

$$a_1 \alpha_2 - a_2 \alpha_1 = 2\pi. \tag{3.20}$$

According to Lemma (3.3) we shall restrict ourselves to the case where

$$a = (\lambda, 0)$$

$$\alpha = \left(0, \frac{2\pi}{\lambda}\right). \tag{3.21}$$

Definition (3.22). A Zak wave is a state $\omega_{\beta\gamma}$ of C.C.R. such that

$$\omega_{\beta\gamma}(\delta_{x,p}) = 0 \quad \text{if} \quad (x, p) \neq \left(n\lambda, \frac{2\pi m}{\lambda}\right)$$

$$\omega_{\beta\gamma}\left(\delta_{n\lambda, \frac{2\pi m}{\lambda}}\right) = \exp\left(i\beta n\lambda + i\gamma \frac{2\pi m}{\lambda}\right) \exp(i\pi mn) \quad \beta \in \left[0, \frac{2\pi}{\lambda}\right[\quad \gamma \in [0, \lambda[\, .$$

Proposition (3.23). *Let $\pi_{\beta\gamma}$, $\mathcal{H}_{\beta\gamma}$, $\Omega_{\beta\gamma}$ be the G.N.S. triplet associated to $\omega_{\beta\gamma}$*

1) $\omega_{\beta\gamma}$ *is pure; $\pi_{\beta\gamma}$ is irreducible.*
2) $\mathcal{H}_{\beta\gamma}$ *is non-separable.*
3) $\forall \beta'$ *and γ', $\omega_{\beta'\gamma'}$ is a vector state of $\pi_{\beta\gamma}$.*
4) $\omega_{\beta\gamma}$ *is disjoint from any Weyl state and any plane wave.*

Proof. 1) is obvious from theorem (2.18) and the maximality of the subgroup.

2) Is a consequence of Corollary (2.19).

3) Let $(x, p), (x', p') \in \mathbb{R}^2$; one has the following obvious calculation:

$$\left(\Omega_{\beta\gamma} \,|\, \pi_{\beta\gamma}(\delta_{x,p}) \, \pi_{\beta\gamma}(\delta_{x',p'}) \, \pi_{\beta\gamma}(\delta_{-x,-p}) \, \Omega_{\beta\gamma}\right)$$

$$= \exp(-i(xp' - px')) \left(\Omega_{\beta\gamma} \,|\, \pi_{\beta\gamma}(\delta_{x',p'}) \, \Omega_{\beta\gamma}\right)$$

$$= 0 \quad \text{if} \quad (x', p') \neq \left(n'\lambda, \frac{2\pi m'}{\lambda}\right)$$

$$= \exp\left(-i\left(x\frac{2\pi m'}{\lambda} - pn'\lambda\right) + i\beta n'\lambda + i\gamma\frac{2\pi m'}{\lambda} + i\pi m'n'\right)$$

$$\text{if} \quad (x', p') = \left(n'\lambda, \frac{2\pi m'}{\lambda}\right)$$

$$= \exp\left(in'\lambda(\beta + p) + i\frac{2\pi m'}{\lambda}(\gamma - x) + i\pi m'n'\right).$$

Hence the result.

Let us remark that there is an orthonormal basis of $\mathcal{H}_{\beta\gamma}$ which is indexed by the points in the rectangle Δ_λ

$$\Delta_\lambda = \left\{(\tilde{x}, \tilde{p}); \, \tilde{x} \in [0, \lambda[, \, \tilde{p} \in \left[0, \frac{2\pi}{\lambda}\right[\right\}.$$

Let us denote by $|\tilde{x}, \tilde{p})_{\beta\gamma}$, $(\tilde{x}, \tilde{p}) \in \Delta_\lambda$, this basis; one has

$$|\tilde{x}, \tilde{p})_{\beta\gamma} = \pi_{\beta\gamma}(\delta_{\tilde{x},\tilde{p}}) \, \Omega_{\beta\gamma} \, .$$

4) Since the $\pi_{\beta\gamma}$'s are irreducible it is enough to prove the non-unitary equivalence of π_{00} to the usual representation and to the presentation induced by the plane waves. This is achieved if one remarks that unitary equivalence conserves the continuity properties of

$$\lambda \to \pi(\delta_{\lambda(x,p)})$$

for the usual representation

$$x \to \pi(\delta_{x,p})$$

for the plane waves.

The $|\tilde{x}, \tilde{p})_{\beta\gamma}$ diagonalize the $\pi_{\beta\gamma}(\delta_{x,p})$ for

$$(x, p) = \left(n\lambda, \frac{2\pi m}{\lambda}\right), \quad (n, m) \in \mathbb{Z}^2 \, ,$$

in the following sense.

Lemma (3.24). *Within the representation* $\pi_{\beta\gamma}$ *one has*

$$\pi_{\beta\gamma}\left(\delta_{n\lambda, \frac{2\pi m}{\lambda}}\right) |\tilde{x}, \tilde{p})_{\beta\gamma}$$

$$= \exp\left(-i\left(n\lambda\tilde{p} - \frac{2\pi m}{\lambda}\tilde{x}\right)\right) \exp\left(i\left(n\lambda\beta + \frac{2\pi m}{\lambda}\gamma + \pi m n\right)\right) |\tilde{x}, \tilde{p})_{\beta\gamma} \, .$$

Zak waves have the same relationship to the Zak $k \cdot q$ representation [11] that the plane waves have to the usual p representation; that is the reason for which we choose the terminology. In what we have done we have considered special closed subgroups of $(\mathbb{R}^2, \xi)$, namely those whose generators are parallel to the axis or rectangular lattices parallel to the axis. As far as the existence and uniqueness, or the general features of invariant states with respect to closed abelian subgroups of $(\mathbb{R}^2, \xi)$ are concerned, this is sufficient to treat the general case as a result of the Lemma (3.3) and the trivial:

Lemma (3.25). *Let ω be a state of $\overline{\Delta(\mathbb{R}^{2N}, \xi)}$ (resp. pure state) invariant with respect to some closed ξ-abelian subgroup of $(\mathbb{R}^{2N}, \xi)\,\Gamma$. Let M be a symplectic transformation of $(\mathbb{R}^{2N}, \xi)$ and τ_M the corresponding $*$-automorphism of $\overline{\Delta(\mathbb{R}^{2N}, \xi)}$, then $\omega \circ \tau_M$ is an invariant state (resp. invariant pure state) with respect to the ξ-abelian subgroup of $(\mathbb{R}^{2N}, \xi)$ $M^{-1}\Gamma = \{M^{-1}a; \, a \in \Gamma\}$.*

Because of the uniqueness theorem of Stone-von-Neumann [8] such a $*$-automorphism is automatically implemented within the usual Schrödinger representation: it does not change the continuity property

of the state. This is no longer the case for the representation we have considered in this section as we shall see. Firstly let us give an obvious result.

Lemma (3.26). *The group of symplectic transformations on* $(\mathbb{R}^2, \xi)$ *is isomorphic to the group of real two by two matrices of determinant one.*

We can give now the following proposition.

Proposition (3.27). *Let* π *be the representation of* $\overline{\Delta(\mathbb{R}^2, \xi)}$ *induced by a pure plane wave (or a Bloch state); let* τ_M *be the* *-automorphism of* $\overline{\Delta(\mathbb{R}^2, \xi)}$ *corresponding to a symplectic transformation* M *of* $(\mathbb{R}^2, \xi)$*; the necessary and sufficient condition for* τ_M *to be unitarily implemented within* π *is that* M *leaves the axis* $(x, 0)$ *invariant or equivalently that* M *is of the form*

$$M = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \quad \begin{matrix} \alpha, \beta \in \mathbb{R} \\ \alpha \neq 0 \end{matrix} \; .$$

Proof. The sufficiency of the previous condition is clear. Indeed let ω be a pure Bloch state defined by the fact that it is invariant with respect to some subgroup $\Gamma_a = \{(na, 0); n \in \mathbb{Z}\}$. It is pure and $x \to \omega(\delta_{x,p})$ is continuous [cf. Theorem (3.13)]; thus from Lemma (3.25) $\omega \circ \tau_M$ is pure and $\omega \circ \tau_M$ is invariant with respect to

$$M^{-1} \Gamma_a = \{M^{-1}(na, 0) = (n\alpha^{-1} a, 0); n \in \mathbb{Z}\} = \Gamma_{\alpha^{-1}a}.$$

Moreover $\omega \circ \tau_M(\delta_{x,p}) = \omega(\delta_{\alpha x + \beta p, \alpha^{-1} p})$ and the function $x \to \omega \circ \tau_M(\delta_{x,p})$ is still continuous; then, from Theorem (3.13) and Proposition (3.19), $\omega \circ \tau_M$ is a vector state of π_0 [cf. Proposition (3.6)]; hence (cf. [6], 2.4.6.) τ_M is unitarily implemented within π_0.

On the other hand assume that M does not leave the axis $(x, 0)$ invariant

$$M(x, 0) = (\alpha x, \gamma x)$$

and moreover that it is unitarily implemented within π_0 so that there exists an unitary operator U on $\mathcal{H}_0$ such that

$$U \pi_0(\delta_{x,p}) U^{-1} = \pi_0(\tau_M \delta_{x,p}) .$$

Let us restrict ourselves to $p = 0$

$$U \pi_0(\delta_{x,0}) U^{-1} = \pi_0(\delta_{\alpha x, \gamma x}) .$$

The left hand side is (weakly) continuous for every x but the right hand side has a discontinuity at $x = 0$ which is a contradiction.

One has already encountered this fact in Proposition (3.9) for the group of x-automorphisms associated to the group of symplectric

transformations

$$M_t = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix} \quad \begin{matrix} t \in \mathbb{R} \\ m > 0 \end{matrix}. \tag{3.28}$$

Finally we are left with the study of relations between the Zak waves corresponding to different lattices. We shall not treat the general case but only the case of a rectangular lattice. The next result shows how the equivalence classes of the representations depend in a non-smooth manner on the shape of the lattice, which is a serious drawback for the physical interpretation of the theory.

Proposition (3.29). *Let ω_1 and ω_2 be two Zak waves corresponding to two different rectangular lattices along the axis of basis a_i respectively; let χ_i the corresponding characters, let π_i, $\mathscr{H}_i$, Ω_i the corresponding G.N.S. triplets; then π_1 is unitarily equivalent to π_2 if and only if the ratio a_1/a_2 is rational.*

Proof. According to Proposition (3.23) we can restrict ourselves to the case where $\chi_i\left(na_i, \dfrac{2\pi m}{a_i}\right) = (-1)^{mn}$ for $m, n \in \mathbb{Z}$. Then the nesessary and sufficient condition for the equivalence of π_1 and π_2 is the existence of a vector $\Phi \in \mathscr{H}_1$ such that

$$\pi_1\left(\delta_{na_2, \frac{2\pi m}{a_2}}\right)\Phi = (-1)^{nm}\Phi \qquad n, m \in \mathbb{Z} \tag{3.30}$$

(see [6], 2.4.6.) since by Theorem (2.18) the vector state ω_Φ is equal to ω_2.

As we noticed previously, Φ can be written

$$\Phi = \sum_{m,n} f(n, m)\, \pi_1\left(\delta_{\tilde{x}_n, \tilde{p}_m}\right)\Omega_1 \tag{3.31}$$

with $\displaystyle\sum_{m,n} |f(n,m)|^2 < \infty$ and $(\tilde{x}_n, \tilde{p}_m) \in \Delta_{a_1}$ and the $\pi_1(\delta_{\tilde{x}_n, \tilde{p}_n})\Omega_1$ linearly independent.

Up to a trivial phase factor, the $\pi_1\left(\delta_{na_2, \frac{2\pi m}{a_2}}\right)$ permute the set of $\pi_1(\delta_{\tilde{x}_n, \tilde{p}_m})\Omega_1$, hence in view of the invariance of Φ and the normalizability of Φ the sum (3.31) has to be finite and given $\tilde{x}_i$ there exist p and q integers such that

$$\tilde{x}_i + p a_2 = \tilde{x}_i + q a_1\,;$$

hence the result

$$a_1/a_2 = p/q\,. \tag{3.32}$$

Consequently we have proved the necessity of (3.32). In order to prove the sufficiency, let us consider the case where $q = 1$ since the

general case can be dealt by a repeated application of this reasoning; the vector Φ which satisfies (3.30) is easy to construct

$$\Phi = \frac{1}{\sqrt{p}} \sum_{j=1}^{p-1} \pi_1 \left(\delta_{\frac{j}{p} a_1, 0} \right) \Omega_1 \,. \tag{3.33}$$

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