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Section A :
Physique théorique.

Gauge transformations of second type and their implementation. II. Bosons

by

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ABSTRACT. — A necessary and sufficient condition for implementation of some local gauge transformations in a class of irreducible representations of the C. C. R.-algebra (« Weyl algebra ») is proved. Not all of the pure states induced by these representations are unitarily equivalent to « physically pure » states; it is shown that a state of the class we consider is unitarily equivalent to a physically pure one if and only if a certain property (characterizing the « discrete » states) holds. Unlike the fermion case, they are quasi-free states which are not discrete. The discrete quasi-free states are all equivalent to the only Fock state of this class.

I. PRELIMINARIES

A. The Problem.

In the following paper we consider gauge transformations of the second type over a free Bose system. More precisely if π is a Weyl representation ([1]) of the C. C. R.-algebra Δ then it is equivalent to deal with a family

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([1]) See further and [*1*] for the definition.

$\{ a_k^+, a_k^- \}_{k\in\mathbb{N}}$ of creation and annihilation operators on an Hilbert space $\mathcal{H}$; the gauge transformations of the second type we consider are

$$a_k^+ \mapsto e^{i\lambda_k\theta}a_k^+, \qquad a_k^- \mapsto e^{-i\lambda_k\theta}a_k^-$$

with $\lambda_k\theta$ on the real line.

Such a transformation is induced by an automorphism τ_θ of the C. C. R.-algebra $\Delta \equiv \overline{\Delta(H, \sigma)}$, which is described in the next paragraph. As in [3] we look for irreducible representations of Δ for which the evolution $\theta \mapsto \tau_\theta$ is implemented by a (strongly) continuous unitary representation of the real line $\theta \mapsto U_\theta$. Such are the head lines of the programme sketched by Dell'Antonio in [4]. We solve fully the problem in the case where the generator of τ_θ is diagonalizable.

B. The Boson C*-algebra and some of its Gauge transformations of second type.

Let (H_0, σ) be a separable symplectic space, i. e. a real vector space equipped with a regular, antisymmetric, real bilinear form, which turns H_0 into a locally convex topological space whose topology is defined by the semi-norms:

$$\rho_\varphi : \psi \mapsto |\sigma(\varphi, \psi)| \qquad \varphi, \psi \in H_0$$

We suppose from now, except mention of the contrary, that H_0 is complete for this topology; we call H_0 *σ-complete*.

Let $\Delta(H_0, \sigma)$ be the algebra generated by finite linear combinations of δ_ψ's, $\psi \in H_0$, such that:

$$\delta_\psi(\varphi) = 0 \quad \text{if} \quad \psi \neq \varphi$$

and

$$\delta_\psi(\psi) = 1$$

with the product law:

$$\delta_\psi\delta_\varphi = e^{-i\sigma(\psi,\varphi)}\delta_{\psi+\varphi}$$

and the involution:

$$\delta_\psi \mapsto \delta_\psi^* = \delta_{-\psi}$$

Let $\mathcal{R}(H_0, \sigma)$ be the set of non-degenerated representations π of $\Delta(H_0, \sigma)$ such that the mapping:

$$\lambda \in \mathbb{R}, \qquad \lambda \mapsto \pi(\delta_{\lambda\psi})$$

is strongly continuous.

Let $\mathcal{F}(H_0, \sigma)$ the set of states of $\Delta(H_0, \sigma)$. We define a norm on $\Delta(H_0, \sigma)$ by:

$$x \in \Delta(H_0, \sigma), \qquad \|x\| = \sup_{\omega\in\mathcal{F}(H_0,\sigma)} \sqrt{\omega(x^*x)}$$

It is a C*-algebra norm [1A].

The closure of $\Delta(H_0, \sigma)$ with respect to this norm will be denoted $\Delta_0 = \overline{\Delta(H_0, \sigma)}$ and we shall call it the C. C. R.-algebra (Some call it the « Weyl algebra » [2]).

Suppose Λ is a densily defined linear operator on $H \subset H_0$ such that:

i) dim (ker Λ) is not odd,

ii) $|\Lambda|$ is a diagonalizable operator in a symplectic base (where $\Lambda = J_0 |\Lambda|$ in the polar decomposition).

We choose a complex structure J of H_0 such that

$$\begin{cases} J \,|\, (\ker \Lambda)^{\perp} = J_0 \,|\, (\ker \Lambda)^{\perp} . \\ J \,|\, \ker \Lambda \quad \text{is an arbitrary complex structure of} \quad \ker \Lambda. \end{cases}$$

We shall write:

$$|\Lambda| = \sum_{k \in \mathbb{N}} \lambda_k P_{H_k}, \qquad \lambda_k \in \mathbb{R}$$

where P_{H_k} are the orthogonal projections on H_k and H_k a two-dimensional real subspace of H, which is invariant by J, such that $H_0 = \bigoplus_{k \in \mathbb{N}} H_k$ and $H = \bigoplus_{k \in \mathbb{N}} H_k$ (From now we denote by $\oplus$ the Hilbert sum and by $\oplus$ the direct sum). We remark that some λ_k are possibly not different.

J defines a σ-permitted hilbertian form s on H_0 (or H)

$$(s(\psi, \varphi) = -\sigma(J\psi, \varphi)) \quad [1].$$

It is with that scalar product we use H_0 as an Hilbert space. Λ is the infinitesimal generator of a one-parameter strongly continuous orthogonal group $\{ T_\theta \}_{\theta \in \mathbb{R}}$ on H_0. By [*1*, (4.1.1)], we can define an automorphism τ_θ of Δ_0 with $\tau_\theta(\delta_\psi) = \delta_{T_\theta \psi}$.

IMPORTANT REMARK.

Let $\Delta = \Delta(H, \sigma) \subseteq \Delta_0$. H *is invariant by* Λ *and* J *therefore* $\tau_\theta \Delta = \Delta$ *and* τ_θ *can be restricted to an automorphism of* Δ. *All arguments and computations in the sequel are about* Δ.

II. THE CLASS OF REPRESENTATIONS WE CONSIDER

Let:

$$\Delta_k \equiv \overline{\Delta(H_k, \sigma)}$$

Let $\pi'_k \in \mathscr{R}(H_k, \sigma)$ be an irreducible representation of Δ_k into the separable Hilbert space $\mathscr{H}_k$. Let ω_k be such that $\omega_k(\delta_\psi) = e^{-\frac{1}{4}s(\psi,\psi)}$ with $\delta_\psi \in \Delta_k$. ω_k is a pure state of Δ_k [*1*, (3.2.1) and (3.2.2)] to which corresponds, in the Gelfand-Naimark-Segal construction, the representation π_k, called the Schrödinger representation, and the cyclic vector $\xi_k \in \mathscr{H}_k$.

It is well-known, since von Neumann [5], that π_k and π'_k are unitarily equivalent, i. e. there exists a unitary operator U_k on $\mathcal{H}_k$ such that

$$\forall x \in \Delta_k \qquad \pi_k(x) = U_k \pi'_k(x) U_k^*$$

Let $\pi = \bigotimes_{k \in \mathbb{N}} \pi_k$ and $\pi' = \bigotimes_{k \in \mathbb{N}} \pi'_k$. π and π' are representations of Δ into $\mathcal{H} = \bigotimes_{k \in \mathbb{N}} \mathcal{H}_k$. Recall that each $\Omega = \bigotimes_{k \in \mathbb{N}} \Omega_k$. Ω_k being a vector of $\mathcal{H}_k$, determines an incomplete tensor product $\mathcal{H}^{\Omega} = \bigotimes_{k \in \mathbb{N}}^{\mathscr{C}(\Omega)} \mathcal{H}_k$, with $\mathscr{C}(\Omega)$ the equivalence class of Ω for the relation $\approx$

$$\left(\Omega \approx \Omega' \quad \text{iff} \quad \sum_{k \in \mathbb{N}} | 1 - (\Omega_k | \Omega'_k) | < +\infty \right)$$

The $\mathcal{H}^{\Omega}$'s are invariant subspaces of π' and the restriction of π' to those subspaces, denoted by π'_{Ω}, are irreducible and therefore π' is the direct sum of the set of the π'_{Ω}.

Let $U = \bigotimes_{k \in \mathbb{N}} U_k$. It is a unitary operator on $\mathcal{H}$ [6, lemma 3.1, def. 3.1]. Clearly:

$$\forall x \in \Delta \qquad \pi(x) = U \pi'(x) U^*$$

So every irreducible subrepresentation π'_{Ω} of π' is unitarily equivalent to the subrepresentation $\pi_{U\Omega}$ of π. Therefore we can restrict our study to the consideration of the irreducible subrepresentations of π.

PROPOSITION II.1 (cf. [3]) (²). — *π_{Ω} is unitarily equivalent to $\pi_{\Omega'}$ if and only if Ω and Ω' are unitarily equivalent.*

Proof. — Recall that $\Omega = \bigotimes_{k \in \mathbb{N}} \Omega_k$ and $\Omega' = \bigotimes_{k \in \mathbb{N}} \Omega'_k$ are weakly equivalent iff $\sum_{k \in \mathbb{N}} (1 - |(\Omega_k | \Omega'_k)|) < +\infty$. Suppose that Ω and Ω' are weakly equivalent. By [6, def. 6.1.1 and lemma 6.1.1], one can find for each $k \in \mathbb{N}$ a $v_k \in \mathbb{R}$ such that

$$(\Omega'_k)_{k \in \mathbb{N}} \approx (e^{iv_k} \Omega_k)_{k \in \mathbb{N}}$$

Let $U = \bigotimes_{k \in \mathbb{N}} e^{iv_k} I_k$. Then $U\Omega \in \mathcal{H}^{\Omega'}$ and we have:

$$\pi_{\Omega'}(x) = U \pi_{\Omega}(x) U^*, \qquad \forall x \in \Delta.$$

(²) This proposition was previously stated by Guichardet [16] for the fermions, and independently by Klauder, McKenna, and Woods [17] for the bosons. We keep our demonstration because of its connection with Powers' methods.

Conversely, if Ω and Ω' are not weakly equivalent, let us denote:

$$\omega_\Omega(x) = (\Omega \mid \pi_\Omega(x)\Omega) \quad , \qquad x \in \Delta$$

and

$$\omega_{\Omega'}(x) = (\Omega' \mid \pi_{\Omega'}(x)\Omega')$$

Let $U_k \in \mathscr{L}(\mathscr{H}_k)$ be a unitary operator such that $U_k\Omega'_k = \Omega_k$ and let

$$\tilde{U}_k = \bigotimes_j {}_1^{k-1} I_j \otimes U_k \otimes \bigotimes_j {}_{k+1}^{\infty} I_j$$

$$u_k = \pi^{-1}(\tilde{U}_k)$$

Let also $E_{n,m} = \bigoplus_k {}_n^m H_k$; $u_{n,m} = \prod_k {}_n^m u_k$, We get:

$$\forall x \in \overline{\Delta(E_{n,m}, \sigma)}, \qquad \omega_{\Omega'}(x) = \omega_\Omega(u_{n,m} x u_{n,m}^*)$$

Let us denote:

$$\omega_{n,m} = \omega_\Omega \mid \overline{\Delta(E_{n,m}, \sigma)}$$

$$\pi_{n,m} = \bigotimes_k {}_n^m \pi_k$$

$$\Omega_{n,m} = \bigotimes_k {}_n^m \Omega_k$$

$$\forall x \in \overline{\Delta(E_{n,m}, \sigma)}, \qquad \omega_{n,m}(x) = (\Omega_{n,m} \mid \pi_{n,m}(x)\Omega_{n,m})$$

As a product of irreducible representations $\pi_{n,m}$ is an irreducible representation [8] hence $\omega_{n,m}$ is a pure state [9, Lemma 2.4] implies that:

$$\| (\omega_\Omega - \omega_{\Omega'}) \mid \overline{\Delta(E_{n,m}, \sigma)} \| = 2(1 - |\omega_{\Omega'}(u_{n,m})|^2)^{\frac{1}{2}}$$
$$= 2\left(1 - \prod_k {}_n^m |(\Omega_k \mid \Omega'_k)|^2\right)^{\frac{1}{2}}$$

Nevertheless:

LEMMA II.1.1 (³). — *Let*

$$\mathscr{N}_n = \bigotimes_k {}_1^n \overline{\Delta(H_k, \sigma)} = \overline{\Delta(E_{1,n}, \sigma)}$$

Then $\Delta = \overline{\bigcup_n \mathscr{N}_n}$. *If* ω_1 *and* ω_2 *are two equivalent pure states of* Δ *then:*

$$\forall \varepsilon > 0 \quad \exists n_0 \quad \textit{such that} \quad n \geqslant n_0 \Rightarrow \| (\omega_1 - \omega_2) \mid \mathscr{N}_n^c \| < \varepsilon$$

We give the proof of this lemma in our Appendix.

(³) We are indebted to R. T. Powers for the proof of Lemma (II.1.1) which is crucial for the sequel of the proof. See also [*18*, Prop. 13] which provides a more general but far less easy proof of Lemma (II.1.1).

Now, $N_n^c = \mathbb{C}_1 \otimes \ldots \otimes \mathbb{C}_n \otimes \bigotimes_{k}{}_{n+1}^{\infty} \Delta_k$, $\mathbb{C}_k = \mathbb{C}I_k$ and $\overline{\Delta(E_{n,m}, \sigma)} \subset N_n^c$.

As $\lim\limits_{m,\infty} \prod_{k}{}_{n}^{m} |(\Omega_k | \Omega'_k)| = 0$ because Ω and Ω' are not weakly equivalent,

$$|| (\omega_\Omega - \omega_{\Omega'}) | \mathcal{N}_n^c || \geqslant \lim_{m,\infty} || (\omega_\Omega - \omega_{\Omega'}) | \overline{\Delta(E_{n,m}, \sigma)} || = 2$$

Hence ω_Ω and $\omega_{\Omega'}$ are not unitarily equivalent. ■

III. THE THEOREM

Let us denote by A_k the field operator, defined by

$$\pi_k(\delta_{\psi_k}) = e^{iA_k(\psi_k)}, \qquad \psi_k \in H_k$$

We shall write the corresponding creation and annihilation operators, as:

$$a^+(\psi_k) = \frac{1}{2}(A_k(\psi_k) - iA_k(J\psi_k))$$

$$a^-(\psi_k) = \frac{1}{2}(A_k(\psi_k) + iA_k(J\psi_k)).$$

$\forall k \in \mathbb{N}$, we choose $\{ \psi_k^1, \psi_k^2 \}$ an orthonormal basis of H_k and we shall use:

$$a_k^+ = a^+(\psi_k^1) \qquad \text{and} \qquad a_k^- = a^-(\psi_k^1).$$

Recall that ξ_k is a cyclic vector corresponding to the state ω_k

$$(\omega_k(\delta_{\varphi_k}) = e^{-\frac{1}{4}s(\varphi_k, \varphi_k)} \qquad \text{for every} \qquad \varphi_k \in H_k)$$

and that $(\xi_k^n)_{n\in\mathbb{N}}$, with $\xi_k^n = \dfrac{1}{\sqrt{n}}(a_k^+)^n\xi_k$, defines an orthonormal basis of $\mathcal{H}_k$.

It follows that the Ω_k 's of Sect. II can be written:

$$\Omega_k = \sum_{n\in\mathbb{N}} \alpha_k^n \xi_k^n \qquad \left(\sum_{n\in\mathbb{N}} |\alpha_k^n|^2 = 1\right)$$

From now, we shall denote $\beta_k^n = |\alpha_k^n|^2$.

A. Statement.

A one-particle evolution τ_θ is implementable for the representation π_Ω if and only if the following condition holds (III.A.1):

$$\sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l \inf(\lambda_k^2(j-l)^2, 1) < +\infty$$

If this occurs, a strongly continuous one-parameter group of unitary operator (we shall call such groups SCOPUG),

$$\{ W_\theta \}_{\theta \in \mathbb{R}}, \qquad W_\theta \in \pi_\Omega(\Delta)'' = \mathscr{L}(\mathscr{H}^\Omega),$$

exists such that:

$$\forall x \in \Delta, \qquad \forall \theta \in \mathbb{R} \qquad \pi_\Omega(\tau_\theta(x)) = W_\theta \pi_\Omega(x) W_{-\theta}$$

B. **Proof.**

B.1. SUFFICIENCY

Suppose

$$\sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l \inf(\lambda_k^2(j-l)^2, 1) < +\infty$$

It is well-known that ([*1*], (4.3) and [*10*], (5.1)):

$$\forall x \in \Delta_k \qquad \pi_k(\tau_\theta(x)) = U_{k,\theta} \pi_k(x) U_{k,\theta}^{-1}$$

with $U_{k,\theta}$ a strongly continuous unitary representation of $\mathbb{R}$ into $\mathscr{H}_k$ such that:

$$U_{k,\theta} = e^{iN_k \lambda_k \theta}$$

with

$$N_k = a^+(\psi_k^1) a^-(\psi_k^1) + a^+(\psi_k^2) a^-(\psi_k^2)$$

where $\psi_k^1 \in H_k$ and $\psi_k^2 = J\psi_k^1$.

Let us build

$$U_\theta = \bigotimes_{k\in\mathbb{N}} U_{k,\theta}.$$

U_θ is a unitary operator on $\mathscr{H}$ [*6*, Lemma 3.1, Def. 3.1].

We get:

$$\forall x \in \Delta \qquad \pi(\tau_\theta(x)) = U_\theta \pi(x) U_\theta^{-1}$$

Changing $U_{k,\theta}$ into $V_{k,\theta} = e^{i\mu_k} U_{k,\theta}$, $\mu_k \in \mathbb{R}$, $V_\theta = \bigotimes_{k\in\mathbb{N}} V_{k,\theta}$ implements τ_θ.

We choose μ_k such that:

$$\forall k \in \mathbb{N} \qquad \operatorname{Arg}(\Omega_k \mid V_{k,\theta}\Omega_k) = 0$$

We get:

$$(\Omega_k \mid V_{k,\theta}\Omega_k)^2 = |(\Omega_k \mid U_{k,\theta}\Omega_k)|^2 = \sum_{(j,l)\in\mathbb{N}^2} \beta_k^j \beta_k^l \cos(2\lambda_k \theta(j-l))$$

Let us consider:

$$\sum_{k\in\mathbb{N}} |1 - (\Omega_k \mid V_{k,\theta}\Omega_k)^2| = \sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l [1 - \cos \lambda_k \theta(j-l)]$$

$$= 2 \sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l \sin^2(\lambda_k \theta(j-l))$$

From our hypothesis

$$\sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l \sin^2(\lambda_k\theta(j-l)) < +\infty$$

for small θ 's,

$$(\Omega \mid V_\theta \Omega) = \prod_{k\in\mathbb{N}} (\Omega_k \mid V_{k,\theta}\Omega_k)$$

converges to a real number different from 0 and $V_\theta \mathscr{H}^\Omega \subset \mathscr{H}^\Omega$. We note now V_θ its restriction to $\mathscr{H}^\Omega$. Hence:

$$\forall x \in \Delta \qquad \pi_\Omega(\tau_\theta(x)) = V_\theta \pi_\Omega(x) V_\theta^*$$

holds. Nevertheless, $\{ V_\theta \}_{\theta\in\mathbb{R}}$ is *not* a group in the general case. A theorem of Kallmann [*11*] provides us the existence of such a SCOPUG $\{ W_\theta \}_{\theta\in\mathbb{R}}$ in $\mathscr{L}(\mathscr{H}^\Omega)$ with:

$$\forall x \in \Delta \quad \forall \theta \in \mathbb{R} \qquad \pi_\Omega(\tau_\theta(x)) = W_\theta \pi_\Omega(x) W_{-\theta}$$

B.2. NECESSITY

Condition (III.A.1) is equivalent to the both following conditions:

(III.B.2.1)
$$\sum_{\substack{(k,j,l)\in\mathbb{N}^3 \\ |\lambda_k|(j-l)\geq 1}} \beta_k^j \beta_k^l < +\infty$$

(III.B.2.2)
$$\sum_{\substack{(k,j,l)\in\mathbb{N}^3 \\ |\lambda_k|(j-l)\leq 1}} \beta_k^j \beta_k^l (j-l)^2 \lambda_k^2 < +\infty$$

Suppose (III.A.1) is false. Then either (III.B.2.1) or (III.B.2.2) is false. Let us recall the two lemmas which prove that in the both cases $\exists\theta \in \mathbb{R}$ such that

$$\sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l \sin^2(\lambda_k\theta(j-l)) = +\infty$$

LEMMA III.B.2.3 (See [*3*, lemma 2.1]). — *Let* $(r_k)_{k\in\mathbb{N}}$, $0 \leq r_k \leq 1$, *and let*

$$\lambda_k \in \mathbb{R}, \qquad |\lambda_k| \geq 1,$$

Then:

$$\left(\sum_{k\in\mathbb{N}} r_k \sin^2(\lambda_k\theta) < +\infty \quad \forall\theta \in I \in \mathscr{V}_{\mathbb{R}}(0)\right) \Rightarrow \sum_{k\in\mathbb{N}} r_k < +\infty$$

Let ν be a bijective enumeration of $\mathbb{N}^3$, $\nu(k, j, l) = m$. Let us write $r_m = \beta_k^j \beta_k^l$ and $\mu_m = \lambda_k(j-l)$. If (III.B.2.1) is false, we get therefore:

$$\sum_{m\in\mathbb{N}} r_m = +\infty \Rightarrow \exists\theta \in \mathbb{R}$$

such that:

$$\sum_{m\in\mathbb{N}} r_m \sin^2(\mu_m\theta) = \sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j\beta_k^l \sin^2(\lambda_k\theta(j-l)) = +\infty .$$

LEMMA III.B.2.4 (See [3, lemma 2.2]). — *If* $f:\mathbb{R}\to\mathbb{R}$, $f(0)=0$, f *differentiable at* 0 *and* $f'(0)=1$, $u_k\in\mathbb{R}$, $(u_k)_{k\in\mathbb{N}}$ *bounded*, $r_k\geqslant 0$, $\forall k\in\mathbb{N}$, *then:*

$$\left(\exists I\in\mathscr{V}_{\mathbb{R}}(0) \quad \text{and} \quad \forall\theta\in I, \ \sum_{k\,1}^{\infty} r_k(f_k(u_k\theta))^2 < +\infty\right) \Leftrightarrow \sum_{k\,1}^{\infty} r_k u_k^2 < +\infty$$

The proof is obvious.

Let us return to the proof of main theorem. Let $\theta\in\mathbb{R}$ such that

$$\sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j\beta_k^l \sin^2(\lambda_k\theta(j-l)) = +\infty$$

Let us denote as in the proof of (II.1):

$$E_{n,m} = \bigoplus_{k\ n}^{m} H_k$$

$$\omega_{n,m} = \omega_\Omega \mid \overline{\Delta(E_{n,m},\sigma)}$$

$$\pi_{n,m} = \bigotimes_{k\ n}^{m} \pi_k$$

$$\Omega_{n,m} = \bigotimes_{k\ n}^{m} \Omega_k$$

$$\mathscr{H}_{n,m} = \bigotimes_{k\ n}^{m} \mathscr{H}_k$$

$$\forall z\in\overline{\Delta(E_{n,m},\sigma)} \qquad \omega_{n,m}(z) = (\Omega_{n,m}\mid\pi_{n,m}(z)\Omega_{n,m})$$

$\pi_{n,m}$ is an irreducible representation, therefore $\omega_{n,m}$ is a pure state. We have:

$$\pi_{n,m}(\tau_\theta(z)) = U_{n,m,\theta}\pi_{n,m}(z)U_{n,m,\theta}^{-1}$$

with

$$U_{n,m,\theta} = \bigotimes_{k\ n}^{m} U_{k,\theta}\,; \qquad U_{k,\theta} = e^{iN_k\lambda_k\theta}$$

N_k is a « number of particles » operator as in (III.B.1).

On the other hand, by a theorem of Glimm and Kadison [12], an $u_{n,m}(\theta)\in\overline{\Delta(E_{n,m},\sigma)}$ exists such that:

$$\omega_{n,m}(\tau_\theta(z)) = \omega_{n,m}(u_{n,m}(\theta)zu_{n,m}^*(\theta))$$

Hence:

$$(U^*_{n,m,\theta}\Omega_{n,m} \mid \pi_{n,m}(z)U^*_{n,m,\theta}\Omega_{n,m}) = (\pi_{n,m}(u^*_{n,m}(\theta))\Omega_{n,m} \mid \pi_{n,m}(z)\pi_{n,m}(u^*_{n,m}(\theta))\Omega_{n,m})$$

and [13, corollary, p. 84]

$$\pi_{n,m}(u^*_{n,m}(\theta))\Omega_{n,m} = e^{i\rho}U^*_{n,m,\theta}\Omega_{n,m}$$

So:

$$|\,\omega_{n,m}(u_{n,m}(\theta))\,| = |\,(\Omega_{n,m} \mid U_{n,m,\theta}\Omega_{n,m})\,| = \prod_n^m |\,(\Omega_k \mid U_{k,\theta}\Omega_k)\,|$$

A theorem of Powers and Størmer [9, lemma 2.4] shows us that:

$$\|\,(\omega_\Omega - \omega_\Omega \circ \tau_\theta) \mid \overline{\Delta(E_{n,m}, \sigma)}\,\| = 2(1 - |\,\omega_\Omega(u_{n,m}(\theta))\,|^2)^{\frac{1}{2}}$$

We apply lemma (II.1.1) with:

$$\mathcal{N}_n = \bigotimes_k{}_1^n \overline{\Delta(H_k, \sigma)}$$

$$\mathcal{N}_n^c = \mathbb{C}_1 \otimes \ldots \otimes \mathbb{C}_n \otimes \bigotimes_k{}_{n+1}^\infty \Delta_k$$

$$\mathbb{C}_k = \mathbb{C}I_k, \qquad 1 \leqslant k \leqslant n.$$

Obviously:

$$\overline{\Delta(E_{n,m}, \sigma)} \subset \mathcal{N}_n^c$$

Therefore:

$$\|\,(\omega_\Omega - \omega_\Omega \circ \tau_\theta) \mid N_n^c\,\| \geqslant \lim_{m,\infty} \|\,(\omega_\Omega - \omega_\Omega \circ \tau_\theta) \mid \overline{\Delta(E_{n,m}, \sigma)}\,\| \geqslant \lim_{m,\infty} 2\left(1 - \prod_{k\,n}^m |\,(\Omega_k \mid U_{k,\theta}\Omega_k)\,|^2\right)^{\frac{1}{2}}$$

Now:

$$\sum_{k\in\mathbb{N}} |\,1 - |\,(\Omega_k \mid U_{k,\theta}\Omega_k)\,|^2\,| = 2 \sum_{(k,j,l)\in\mathbb{N}^3} \beta_k^j \beta_k^l \sin^2(\lambda_k\theta(j-l)) = +\infty$$

Therefore:

$$\lim_{m,\infty} \prod_{k\,n}^m |\,(\Omega_k \mid U_{k,\theta}\Omega_k)\,|^2 = 0$$

and:

$$\forall n \in \mathbb{N} \qquad \|\,(\omega_\Omega - \omega_\Omega \circ \tau_\theta) \mid \mathcal{N}_n^c\,\| = 2$$

So, lemma (II.1.1) enables us to assert that ω_Ω and $\omega_\Omega \circ \tau_\theta$ are not unitarily equivalent; hence there is no unitary operator $U_\theta \in \mathcal{L}(\mathcal{H}^\Omega)$ such that:

$$\forall x \in \Delta \qquad \pi_\Omega(\tau_\theta(x)) = U_\theta \pi_\Omega(x) U_\theta^*$$

τ_θ is not implementable for the representation π_Ω. ■

IV. OTHER PROPOSITIONS AND REMARKS

1.

$$\mathcal{N}_{\Omega}^{\Lambda} = \left\{ \theta \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{There exists a unitary operator} \\ \mathrm{U}_{\theta} \in \mathscr{L}(\mathscr{H}^{\Omega}) \quad \text{such that} \\ \forall x \in \Delta \quad \pi_{\Omega}(\tau_{\theta}(x)) = \mathrm{U}_{\theta}\pi_{\Omega}(x)\mathrm{U}_{\theta}^{*} \end{array} \right\}$$

is an additive subgroup of $\mathbb{R}$ [3, IV.2].

2. If

$$\sum_{\substack{(k,j,l)\in\mathbb{N}^3 \\ j\neq l}} \beta_k^j \beta_k^l < +\infty$$

We shall say that representation π_{Ω} is a *discrete* one. Theorem (III.A) implies that every one-particle evolution is implementable for all the discrete representations. The corresponding state ω_{Ω} will be too called a *discrete* one.

3. We have *not* the corresponding property of [*3*, (IV.3.1)] to conclude that, if π_{Ω} is not a discrete representation and if $\{ \lambda_k \}_{k\in\mathbb{N}}$ has neither 0 non infinite as accumulation points, then $\mathcal{N}_{\Omega}^{\Lambda} = a\mathbb{Z}$, $a \in \mathbb{R}_+$ ($\mathbb{Z}$ the additive group of the relative integers) because $(\mu_m = \lambda_k(j - l))_{m\in\mathbb{N}}$ can have ∞ as limit point even if $\{ \lambda_k \}_{k\in\mathbb{N}}$ does not. Cf. [*4*].

4. Physically pure states, quasi-free states and connected questions.

4.1. Definition. — A state ω_{Ω} defined by

$$\Omega = \bigotimes_{k\in\mathbb{N}} \Omega_k, \qquad \Omega_k = \sum_{n\in\mathbb{N}} \alpha_k^n \xi_k^n$$

will be called a « physically pure » one iff $\alpha_k^n = 0 \ \forall n \neq m(k)$.

4.2. Proposition. — *There exists a physically pure state $\omega_{\Omega'}$ unitarily equivalent to ω_{Ω} iff ω_{Ω} is a discrete state.*

Proof. — Suppose ω_{Ω} is unitarily equivalent to a physically pure state $\omega_{\Omega'}$ with

$$\Omega' = \bigotimes_{k\in\mathbb{N}} \Omega'_k, \qquad \Omega'_k = e^{i\rho_k}\xi_k^{m(k)}, \qquad \forall k \in \mathbb{N}$$

Recall that ω_{Ω} and $\omega_{\Omega'}$ are unitarily equivalent iff (II.1):

$$\sum_{k\in\mathbb{N}} (1 - |(\Omega_k \mid \Omega'_k)|^2) < +\infty$$

hence:

$$\sum_{k\in\mathbb{N}}(1-|\alpha_k^{m(k)}|^2)=\sum_{k\in\mathbb{N}}(1-\beta_k^{m(k)})<+\infty$$

Now,

$$\sum_{\substack{j,l\\ j\neq l}}\beta_k^j\beta_k^l=\sum_{\substack{j,l\\ j\neq l\\ j,l\neq m(k)}}\beta_k^j\beta_k^l+2\sum_{n\neq m(k)}\beta_k^n$$

and:

$$\sum_{\substack{j,l\\ j\neq l\\ j,l\neq m(k)}}\beta_k^j\beta_k^l\leqslant\Big(\sum_{n\neq m(k)}\beta_k^n\Big)^2=(1-\beta_k^{m(k)})^2$$

So:

$$\sum_{\substack{j,l\\ j\neq l}}\beta_k^j\beta_k^l\leqslant(1-\beta_k^{m(k)})^2+2(1-\beta_k^{m(k)})$$

and:

$$\sum_{\substack{(k,j,l)\in\mathbb{N}^3\\ j\neq l}}\beta_k^j\beta_k^l<+\infty$$

i. e., ω_Ω is a discrete state.

Conversely, if

$$\sum_{\substack{(k,j,l)\in\mathbb{N}^3\\ j\neq l}}\beta_k^j\beta_k^l<+\infty$$

$$\sum_{\substack{j,l\\ j\neq l}}\beta_k^j\beta_k^l=1-\sum_{n\in\mathbb{N}}(\beta_k^n)^2=\sum_{n\in\mathbb{N}}(\beta_k^n-\beta_k^{n^2})=\sum_{n\in\mathbb{N}}\beta_k^n(1-\beta_k^n)$$

$$\sum_{\substack{(k,j,l)\in\mathbb{N}^3\\ j\neq l}}\beta_k^j\beta_k^l=\sum_{(k,n)\in\mathbb{N}^2}\beta_k^n(1-\beta_k^n)<+\infty$$

Let:

$$\mathrm{M}_k=\left\{n\in\mathbb{N}\mid\beta_k^n>\frac{1}{2}\right\}$$

$$\mathrm{M}=\bigcup_{k\in\mathbb{N}}(\{k\}\times\mathrm{M}_k)$$

$$\mathrm{L}=\mathbb{N}\times\mathbb{N}-\mathrm{M}\,.$$

Then:

$$\sum_{(k,n)\in\mathrm{M}}(1-\beta_k^n)<+\infty$$

$$\sum_{(k,n)\in\mathrm{L}}\beta_k^n<+\infty$$

Now $L_0 = \{ k \mid M_k = \emptyset \}$ has to be finite, because $\sum_{n \in \mathbb{N}} \beta_k^n = 1$ and:

$$\sum_{k \in L_0} \sum_{n \in \mathbb{N}} \beta_k^n = \text{Card } L_0 \leqslant \sum_{(k,n) \in L} \beta_k^n < +\infty$$

In each M_k we can choose an $m(k)$ and we have:

$$\sum_{k \in \mathbb{N}} (1 - \beta_k^{m(k)}) \leqslant \sum_{(k,n) \in M} (1 - \beta_k^n) < +\infty$$

We can take:

$$\Omega'_k = \xi_k^{m(k)}, \qquad \Omega' = \bigotimes_{k \in \mathbb{N}} \Omega'_k$$

to see that:

$$\sum_{k \in \mathbb{N}} (1 - |(\Omega_k \mid \Omega'_k)|) = \sum_{k \in \mathbb{N}} (1 - \sqrt{\beta_k^{m(k)}}) < +\infty$$

and so a physically pure state $\omega_{\Omega'}$ is unitarily equivalent to ω_Ω. ■

4.3. LEMMA. — *Let*

$$\omega_\Omega = \bigotimes_{k \in \mathbb{N}} \omega_{\Omega_k},$$

then ω_Ω *is a Fock state* $\Leftrightarrow$ ω_{Ω_k} *is a Fock state* $\forall k \in \mathbb{N}$.

Proof. — Let ω_Ω be a Fock state, ω_Ω is a primary state; hence [*14*]:

$$\omega_\Omega(\delta_\varphi) = e^{-\frac{1}{4}s'(\varphi,\varphi)}$$

with s' a σ-allowed hilbertian structure on H.

If $\varphi \in H_k$, a real scalar product s_k exists on H_k such that:

$$\omega_\Omega(\delta_\varphi) = e^{-\frac{1}{4}s_k(\varphi,\varphi)} \qquad \text{and} \qquad s_k = -\sigma \circ J_k$$

J_k the only complex structure on H_k such that s_k turns out to be non negative ($J\psi_k^1 = \psi_k^2$). Therefore for every $k \in \mathbb{N}$, ω_{Ω_k} is *the* Fock state on $\overline{\Delta(H_k, \sigma)}$. Conversely, if ω_{Ω_k} is the only Fock state in $\Delta_k = \overline{\Delta(H_k, \sigma)}$ for every $k \in \mathbb{N}$, $\varphi_k \in H_k$, $\omega_\Omega(\delta_{\varphi_k}) = e^{-\frac{1}{4}s_k(\varphi_k, \varphi k)}$, $s_k = -\sigma \circ J_k$. We take J a complex structure of H such that $J \mid H_k = J_k$ and we get $\omega_\Omega(\delta_\varphi) = e^{-\frac{1}{4}s(\varphi,\varphi)}$ $\forall \varphi \in H$ with $s = -\sigma \circ J$.

4.4. COROLLARY. — *Among the states of the type* ω_Ω *there is only one Fock state.*

Let ω_Ω be a physically pure state;

$$\Omega = \bigotimes_{k \in \mathbb{N}} \Omega_k$$

$\Omega_k = \xi_k^{m(k)}$. Then $\forall \varphi \in H_k$

$$\omega_\Omega(\delta_\varphi) = e^{-\frac{1}{2}\|\varphi\|^2} \sum_{p}{}_0^{m(k)} \frac{(-1)^p}{(m(k)-p)!\,p!^2} \|\varphi\|^{2p}$$

$$= \exp\left(-\frac{1}{2}\|\varphi\|^2\right) L_{m(k)}(\|\varphi\|^2)$$

$L_{m(k)}$ being the Laguerre polynomial of degree $m(k)$ as an easy computation shows.

The only Fock state of the type ω_Ω is constructed with $\Omega_k = \xi_k \ \forall k \in \mathbb{N}$. The ω_Ω 's unitarily equivalent to the Fock state are such that

$$\sum_{k \in \mathbb{N}} (1 - \beta_k^0) < +\infty \quad \left(\xi_k^0 = \xi_k,\ \beta_k^0 = |\alpha_k^0|^2,\ \Omega_k = \sum_{n \in \mathbb{N}} \alpha_k^n \xi_k^n \right). \ \blacksquare$$

4.5. DEFINITION. — A quasi-free state on Δ is a state ω for which

$$\omega(\delta_\varphi) = e^{-\frac{1}{2}s'(\varphi,\varphi) + i\chi(\varphi)}$$

with s' a σ-allowed hilbertian structure on H and χ in the algebraic dual of H.

4.6. COROLLARY. — *Let ω_Ω be a quasi-free state and*

$$c_k \in \mathbb{C}, \qquad |c_k| = (\chi(\psi_k^1)^2 + \chi(\psi_k^2)^2)^{\frac{1}{2}}$$

the following assertions are equivalent:

i) $\sum_{k \in \mathbb{N}} |c_k|^2 < +\infty$.

ii) ω_Ω is a discrete state.

iii) ω_Ω is unitarily equivalent to the Fock state $\omega_{\bigotimes_{k \in \mathbb{N}} \xi_k} = \omega_s$.

Proof. — *iii)* $\Rightarrow$ *ii)* is obvious by Proposition (4.2).

i) $\Rightarrow$ *iii)*

$$\omega_\Omega(\delta_\varphi) = \exp\left[-\frac{1}{2}s'(\varphi,\varphi) + i\chi(\varphi)\right]$$

$$\omega_\Omega = \omega_{s'} \circ \zeta_\chi \quad \text{with} \quad \omega_{s'}(\delta_\varphi) = \exp\left[-\frac{1}{2}s'(\varphi,\varphi)\right]$$

and $\zeta_\chi(\delta_\varphi) = e^{i\chi(\varphi)}\delta_\varphi$. ω_Ω is pure, hence $\omega_{s'}$ is pure and so is the Fock state ω_s [*15*].

We can easily see that

$$\Omega_k = \exp\left(-\frac{|c_k|^2}{2}\right) \sum_{n \in \mathbb{N}} \frac{(c_k)^n}{\sqrt{n!}} \xi_k^n$$

Indeed:

$$\begin{aligned}(\Omega \mid e^{iA(\varphi)}\Omega) &= (\Omega \mid e^{i(a^+(\varphi)+a^-(\varphi))}\Omega) \\ &= e^{-\frac{1}{2}s(\varphi,\varphi)}(e^{-ia^-(\varphi)}\Omega \mid e^{ia^-(\varphi)}\Omega) \\ &= e^{-\frac{1}{2}s(\varphi,\varphi)}e^{is(\sum_1^\infty(\operatorname{Re} c_k\psi_k^1 + \operatorname{Im} c_k\psi_k^2),\varphi)}\end{aligned}$$

If

$$\sum_{k\in\mathbb{N}} |c_k|^2 < \infty, \qquad \chi = \sum_{k}^{\infty}{}_{1} (\operatorname{Re} c_k\psi_k^1 + \operatorname{Im} c_k\psi_k^2)$$

is continuous. So [*1*, (4.4.4)] is unitarily equivalent to the Fock state ω_s.

ii) $\Rightarrow$ *i*)

If ω_Ω is a discrete quasi-free state, we have

$$\Omega_k = \sum_{n\in\mathbb{N}} \alpha_k^n \xi_k^n; \qquad \alpha_k^n = \frac{e^{-\frac{|c_k|^2}{2}}(c_k)^n}{\sqrt{n!}}$$

and $\sum_{k\in\mathbb{N}} (1 - \beta_k^{m(k)}) < \infty$ for a certain $(m(k))_{k\in\mathbb{N}}$. Now, for $n \geqslant 1$

$$\left| \exp\left(-\frac{|c_k|^2}{2}\right).(c_k)^n/\sqrt{n!} \right| \leqslant n^{\frac{n}{2}}e^{-\frac{n}{2}}/\sqrt{n!} \leqslant (2\pi)^{-\frac{1}{4}} < 1\,.$$

Therefore $m(k) = 0\ \forall k \in \mathbb{N} - \mathrm{L}$, L finite and $\sum_{k\in\mathbb{N}} (1 - \beta_k^0) < \infty$ which implies that $\prod_{k\in\mathbb{N}} \exp(-|c_k|^2/2)$ converges and is different from 0. In other words:

$$\sum_{k\in\mathbb{N}} |c_k|^2 < \infty\,. \quad \blacksquare$$

4.7. Remark. — In the opposite of the fermion case [*3*, IV.4.3] there are non discrete quasi-free states; they are constructed with χ no continuous.

APPENDIX

LEMMA II.1.1. — *Let*

$$\mathcal{N}_n = \bigotimes_{k}^{n} {}_1 \overline{\Delta(\mathrm{H}_k, \sigma)}, \qquad \text{then} \qquad \Delta = \overline{\bigcup_{n \in \mathbb{N}} \mathcal{N}_n}.$$

If ω_1 *and* ω_2 *are two unitarily equivalent pure states of* Δ, *then:*

$$\lim_{n,\infty} \| (\omega_1 - \omega_2) \mid \mathcal{N}_n^c \| = 0 .$$

Proof (R. T. Powers). — By [*12*], if ω_1 and ω_2 are unitarily equivalent, there exists an $u \in \Delta$ such that $uu^* = u^*u = \mathrm{I}_\Delta$ and $\forall x \in \Delta$, $\omega_1(x) = \omega_2(u^*xu)$. Let $1 > \varepsilon > 0$. $\exists n \in \mathbb{N}$, $\exists b \in \mathcal{N}_n$ with $\| b - u \| < \varepsilon$. Since $\| b - u \| < \varepsilon$, b^{-1} exists. Let $u' = b(b^*b)^{-\frac{1}{2}}$. Then $u' \in \mathcal{N}_n$ and $u'^*u' = u'u'^* = \mathrm{I}_\Delta$. And

$$\begin{aligned} \| u' - u \| &\leqslant \| b(b^*b)^{-\frac{1}{2}} - b \| + \| b - u \| \\ &\leqslant \| b \| \quad \| (b^*b)^{-\frac{1}{2}} - \mathrm{I}_\Delta \| + \varepsilon \end{aligned}$$

Now if $\| y - \mathrm{I}_\Delta \| < 1$:

$$\| y^{-1} - \mathrm{I}_\Delta \| = \left\| \sum_{n}^{\infty} {}_1 (\mathrm{I}_\Delta - y)^n \right\| \leqslant \frac{\| y - \mathrm{I}_\Delta \|}{1 - \| y - \mathrm{I}_\Delta \|}$$

and, for any $\varepsilon' > 0$, one can choose $\varepsilon > 0$ such that $\| (bb^*)^{\frac{1}{2}} - \mathrm{I}_\Delta \| < \varepsilon'$ because $y \mapsto (yy^*)^{\frac{1}{2}}$ is continuous. So:

$$\| u' - u \| \leqslant \| b \| \frac{\varepsilon'}{1 - \varepsilon'} + \varepsilon = \varepsilon''$$

Let ω', such that:

$$\omega_1'(x) = \omega_2(u'^*xu')$$

$$\begin{aligned} \| \omega_1 - \omega_1' \| &= \sup_{\substack{x \in \Delta \\ \|x\| = 1}} | \omega_1(x) - \omega_1'(x) | \\ &= \sup_{\substack{x \in \Delta \\ \|x\| = 1}} | \omega_2(\omega^*xu - u'^*xu') | \\ &\leqslant \sup_{\substack{x \in \Delta \\ \|x\| = 1}} \| u^*xu - u'^*xu + u'^*xu - u'^*xu' \| \\ &\leqslant 2 \| u - u' \| \leqslant 2\varepsilon'' \end{aligned}$$

Now:

$$\omega_2 \mid \mathcal{N}_n^c = \omega_1' \mid \mathcal{N}_n^c$$

because, for $y \in \mathcal{N}_n^c$:

$$\omega_1'(y) = \omega_2(u'^*yu') = \omega_2(y)$$

Hence:

$$\| (\omega_2 - \omega_1) \mid \mathcal{N}_n^c \| = \| (\omega_1' - \omega_1) \mid \mathcal{N}_n^c \| \leqslant 2\varepsilon'' . \quad \blacksquare$$

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REFERENCES

[*1*] J. MANUCEAU, *Ann. Inst. Henri Poincaré*, Vol. VIII, n° 2, 1968, p. 139-161.

[*1A*] J. MANUCEAU, M. SIRUGUE, D. TESTARD and A. VERBEURE, *Commun. Math. Phys.*, t. **32**, 1973, p. 231-243.

[2] A. van Daele and A. Verbeure, *Commun. Math. Phys.*, t. **20**, 1971, p. 268-298.
[3] J. F. Gille and J. Manuceau, *J. Math. Phys.*, t. **13**, 1972, p. 2002.
[4] G. F. Dell'Antonio, *J. Math. Phys.*, t. **12**, 1971, p. 148.
[5] J. von Neumann, « Die Eindentigkeit der Schrödingerschen Operatoren ». *Math. Ann.*, t. **104**, 1931, p. 570.
[6] K. Nakagami, *Kōdai Math. Sem. Rep.*, Vol. **22**, n° 3, 1970, p. 341-354.
[7] J. von Neumann, *On Infinite Direct Products* in Collected Works, Pergamon, New York, 1961.
[8] See, for instance: R. T. Powers, Princeton Thesis, 1967.
[9] R. T. Powers and E. Størmer, *Commun. Math. Phys.*, t. **16**, 1970, p. 1.
[10] J. Manuceau, *Ann. Inst. Henri Poincaré*, Vol. III, n° 2, 1968, p. 117-138.
[11] R. R. Kallman, *Journ. of Functional Analysis*, t. **7**, 1971, p. 43-60.
[12] J. C. Glimm and R. V. Kadison, « Unitary Operators in C*-algebras ». *Pacific J. Math.*, t. **10**, 1960, p. 547-548.
[13] G. G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. John Wiley and Sons Inc., 1972.
[14] J. Manuceau and A. Verbeure, *Commun. Math. Phys.*, t. **9**, 1968, p. 293-302.
[15] J. Manuceau, F. Rocca and D. Testard, *Commun. Math. Phys.*, t. **12**, 1968, p. 43-57.
[16] A. Guichardet, *Ann. Sci. École Normale Sup.*, t. **83**, 1966, p. 1-52.
[17] J. R. Klauder, J. McKenna and E. J. Woods, *J. Math. Phys.*, t. **7**, 1966, p. 822-828.
[18] R. Haag, R. V. Kadison, D. Kastler, *Commun. Math. Phys.*, t. **16**, 1970, p. 81-104.

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